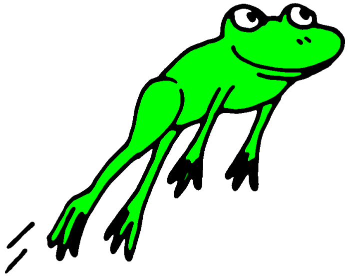
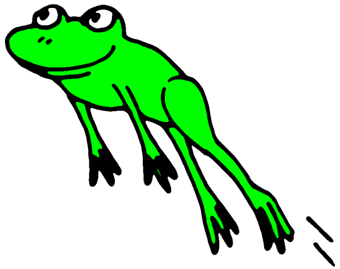
An introduction to deep inference



Victoria Barrett

Inria Saclay, Équipe Partout



Proof Society 2026, Aussois

What is deep inference?

Deep inference is a methodology for the design of proof formalisms.

Originally developed by Alessio Guglielmi to allow for a proof-theoretic treatment of sequential composition.

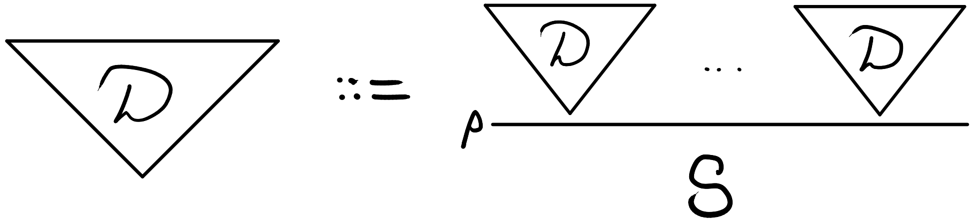
Emphasises:
- less syntax and bureaucracy,
- underlying geometric nature of proofs,
- regularity of inference rules and normalisation procedures in different systems

What is a proof formalism?

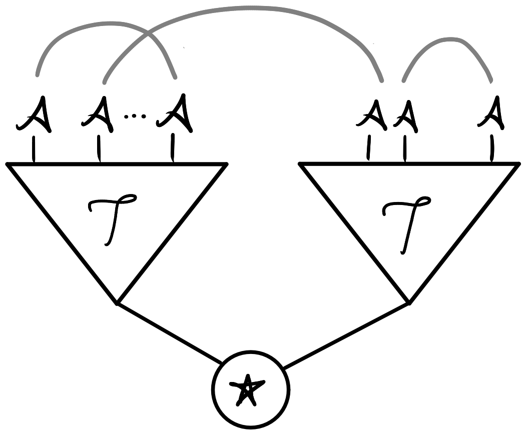
Sequent Calculus

$$\mathcal{F} ::= \mathcal{A} \mid \mathcal{U} \mid \mathcal{F} \odot \mathcal{F} \mid \star\mathcal{F} \mid \cdots$$

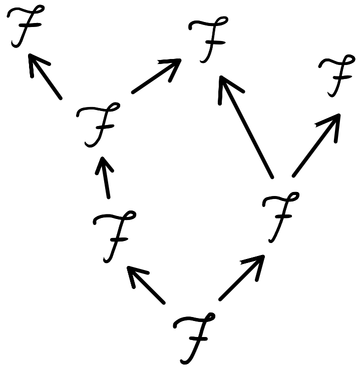
$$\mathcal{S} ::= \mathcal{F}, \ldots, \mathcal{F} \vdash \mathcal{F}, \ldots, \mathcal{F}$$



Proof Nets



Hilbert systems

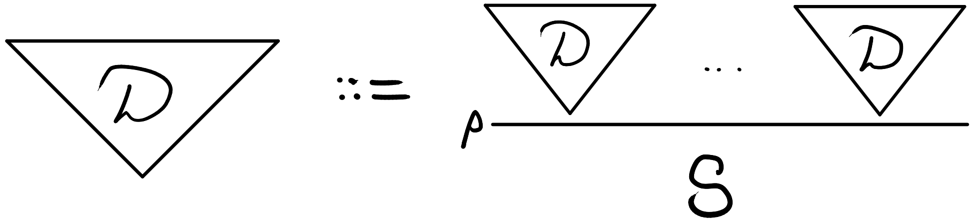


What is a proof formalism?

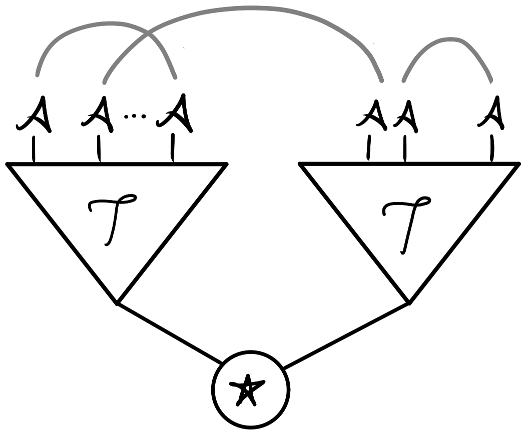
1-sided Sequent Calculus

$$\mathcal{F} ::= \mathcal{A} \mid \mathcal{U} \mid \mathcal{F} \odot \mathcal{F} \mid \star\mathcal{F} \mid \cdots$$

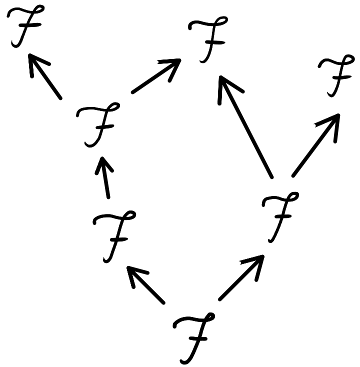
$$\mathcal{S} ::= \cancel{\mathcal{F}, \ldots, \mathcal{F}} \vdash \mathcal{F}, \ldots, \mathcal{F}$$



Proof Nets



Hilbert systems

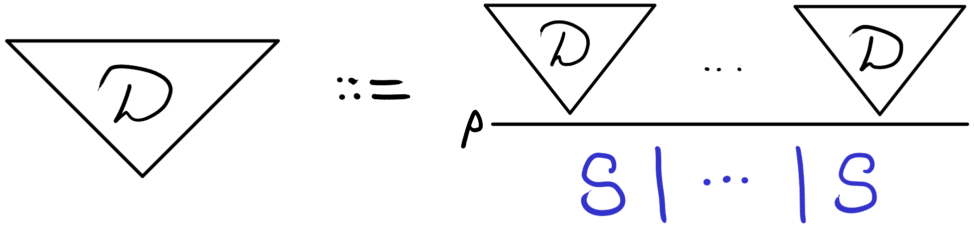


What is a proof formalism?

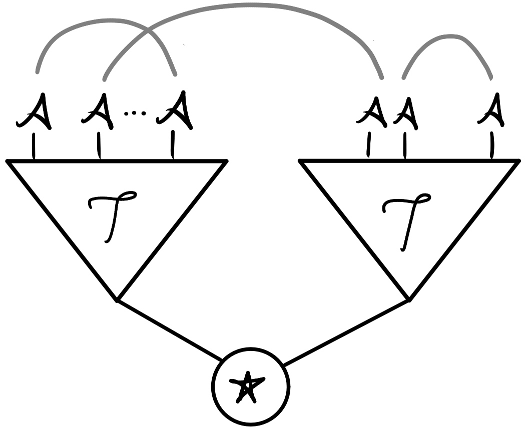
hyper-Sequent Calculus

$$\mathcal{F} ::= \mathcal{A} \mid \mathcal{U} \mid \mathcal{F} \odot \mathcal{F} \mid \star\mathcal{F} \mid \cdots$$

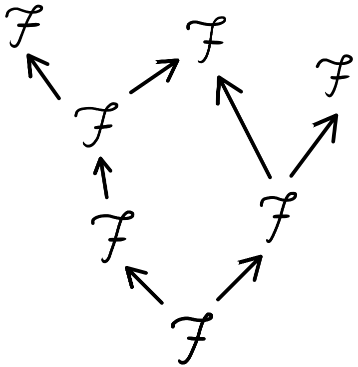
$$S ::= \mathcal{F}, \dots, \mathcal{F} \vdash \mathcal{F}, \dots, \mathcal{F}$$



Proof Nets



Hilbert systems



What makes a formalism good?

Normalisation : proofs have useful normal forms and normalisation procedures are simple

Complexity : theorems have small proofs

Semantics : proofs are represented with little syntactic bureaucracy and it's easy to tell when two proofs are the same

Proof search : it's quick to prove or refute formulae

Expressibility : systems can be given for interesting logics

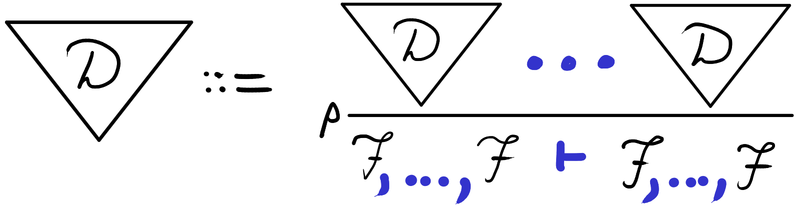
⋮

Towards open deduction, a deep inference formalism

Sequent Calculus

$$\mathcal{F} ::= \mathcal{A} \mid \mathcal{U} \mid \mathcal{F} \odot \mathcal{F} \mid \star\mathcal{F} \mid \cdots$$

formula-level connectives



derivation-level connectives

Inference rules have shallow access to connectives

For example,

$$\text{contR}\,\frac{\Gamma \vdash A, A, \Delta}{\Gamma \vdash A, \Delta} \qquad \text{not} \qquad \frac{\Gamma \vdash A \vee A, \Delta}{\Gamma \vdash A, \Delta}$$

and

$$\wedge\,\frac{\Gamma \vdash A, \Delta \qquad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta}$$

not

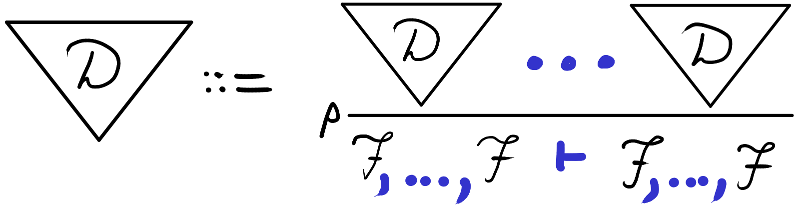
$$\frac{\Gamma \vdash A \vee C, \Delta \qquad \Gamma \vdash B \vee C, \Delta}{\Gamma \vdash (A \wedge B) \vee C, \Delta}$$

Towards open deduction, a deep inference formalism

Sequent Calculus

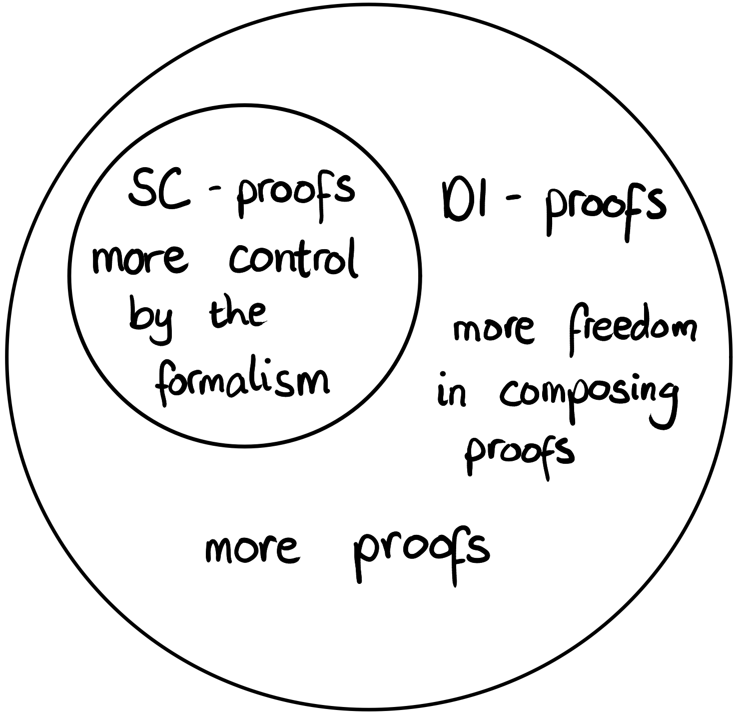
$$\mathcal{F} ::= \mathcal{A} \mid \mathcal{U} \mid \mathcal{F} \odot \mathcal{F} \mid \star\mathcal{F} \mid \cdots$$

formula-level connectives



$$\bigtriangledown\mathcal{D} ::= \rho\,\frac{\bigtriangledown\mathcal{D} \quad \cdots \quad \bigtriangledown\mathcal{D}}{\mathcal{F},\ldots,\mathcal{F} \vdash \mathcal{F},\ldots,\mathcal{F}}$$

derivation-level connectives



SC - proofs

more control by the formalism

DI - proofs

more freedom in composing proofs

more proofs

Inference rules have shallow access to connectives

Towards open deduction, a deep inference formalism

Sequent Calculus

$$\mathcal{F} ::= \mathcal{A} \mid \mathcal{U} \mid \mathcal{F} \odot \mathcal{F} \mid \star\mathcal{F} \mid \cdots$$

formula-level connectives

$$\overset{\mathcal{D}}{\bigtriangledown} ::= \rho\,\dfrac{\overset{\mathcal{D}}{\bigtriangledown}\ \cdots\ \overset{\mathcal{D}}{\bigtriangledown}}{\mathcal{F},\ldots,\mathcal{F} \vdash \mathcal{F},\ldots,\mathcal{F}}$$

derivation-level connectives

Inference rules have shallow access to connectives

Open deduction

$$\boxed{\mathcal{D}} ::= \mathcal{A} \mid \mathcal{U} \mid \boxed{\mathcal{D}} \odot \boxed{\mathcal{D}} \mid \star\boxed{\mathcal{D}} \mid \rho\,\dfrac{\boxed{\mathcal{D}}}{\boxed{\mathcal{D}}}$$

$$\mathcal{F} \subseteq \mathcal{D}$$

one level of connective

Inference rules have deep access to formulae

Towards open deduction, a deep inference formalism

Sequent Calculus

$$\mathcal{F} ::= \mathcal{A} \mid \mathcal{U} \mid \mathcal{F} \odot \mathcal{F} \mid \star\mathcal{F} \mid \cdots$$

formula-level connectives

$$\overset{\mathcal{D}}{\bigtriangledown} ::= \rho\,\frac{\overset{\mathcal{D}}{\bigtriangledown}\ \cdots\ \overset{\mathcal{D}}{\bigtriangledown}}{\mathcal{F},\ldots,\mathcal{F} \vdash \mathcal{F},\ldots,\mathcal{F}}$$

derivation-level connectives

Open deduction

$$\boxed{\mathcal{D}} ::= \mathcal{A} \mid \mathcal{U} \mid \boxed{\mathcal{D}} \odot \boxed{\mathcal{D}} \mid \star\boxed{\mathcal{D}} \mid \rho\,\frac{\boxed{\mathcal{D}}}{\boxed{\mathcal{D}}}$$

$$\mathcal{F} \subseteq \mathcal{D}$$

one level of connective

$$\text{contR}\,\frac{B, C \vdash A, A, D}{B, C \vdash A, D} \longrightarrow \frac{(B\wedge C)\to(A\vee A\vee D)}{(B\wedge C)\to(A\vee D)} \longrightarrow B^{\perp} \vee C^{\perp} \vee \frac{A\vee A}{A} \vee D$$

Towards open deduction, a deep inference formalism

Open deduction

$$\boxed{\mathcal{D}} ::= \mathcal{A} \mid \mathcal{U} \mid \boxed{\mathcal{D}} \odot \boxed{\mathcal{D}} \mid \star \boxed{\mathcal{D}} \mid \rho\,\frac{\boxed{\mathcal{D}}}{\boxed{\mathcal{D}}}$$

$$\mathcal{F} \subseteq \mathcal{D}$$

one level of connective

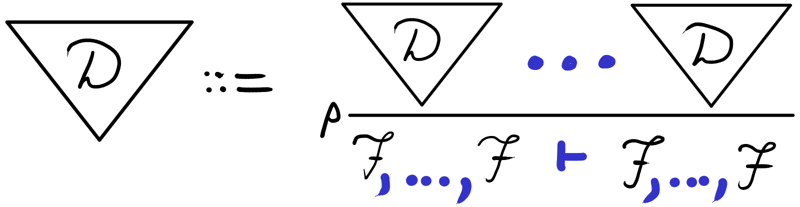
$$\frac{E}{\dfrac{A}{B} \wedge \dfrac{C}{D}} \vee \frac{\dfrac{F \vee G}{H} \wedge J}{K} \qquad \begin{array}{c} \xrightarrow{\text{premise}} \\ \\ \xrightarrow[\text{conclusion}]{} \end{array} \qquad \begin{array}{c} E \vee ((F \vee G) \wedge J) \\ \| \\ (B \wedge D) \vee K \end{array}$$

Towards open deduction, a deep inference formalism

Sequent Calculus

$$\mathcal{F} ::= \mathcal{A} \mid \mathcal{U} \mid \mathcal{F} \odot \mathcal{F} \mid \star\mathcal{F} \mid \cdots$$

formula-level connectives



derivation-level connectives

Open deduction

$$\boxed{\mathcal{D}} ::= \mathcal{A} \mid \mathcal{U} \mid \boxed{\mathcal{D}} \odot \boxed{\mathcal{D}} \mid \star\boxed{\mathcal{D}} \mid \rho\frac{\boxed{\mathcal{D}}}{\boxed{\mathcal{D}}}$$

$$\mathcal{F} \subseteq \mathcal{D}$$

one level of connective

$$\text{cut}\,\frac{\Gamma \vdash A, \Delta \qquad \Xi, A \vdash \Upsilon}{\Gamma, \Xi \vdash \Delta, \Upsilon} \longrightarrow \mathsf{s}\,\frac{\begin{array}{c} \bigwedge_{\Gamma} \gamma \\ \Vert \\ A \vee \bigvee_{\Delta} \delta \end{array} \wedge \bigwedge_{\Xi} \xi}{\bigvee_{\Delta} \delta \vee \begin{array}{c} A \wedge \bigwedge_{\Xi} \xi \\ \Vert \\ \bigvee_{\Upsilon} \upsilon \end{array}}$$

Towards open deduction, a deep inference formalism

$$\mathrm{id}\,\dfrac{}{\vdash a, a^{\perp}} \quad\longrightarrow\quad \mathrm{i}{\downarrow}\,\dfrac{1}{a \parr a^{\perp}} \qquad\qquad \Gamma \vdash \Delta \quad\rightarrow\quad \bigparr_{\gamma\in\Gamma}\gamma^{\perp} \parr \bigparr_{\delta\in\Delta}\delta$$

$$\otimes\,\dfrac{\vdash A, \Gamma \qquad \vdash B, \Delta}{\vdash A \otimes B, \Gamma, \Delta} \quad\longrightarrow\quad {\scriptstyle s}\,\dfrac{(A \parr C) \otimes (B \parr D)}{{\scriptstyle s}\,\dfrac{(A \otimes (B \parr D))}{(A \otimes B) \parr D} \;\parr\; C}$$

We only want inference rules ${\scriptstyle \rho}\dfrac{A}{B}$ that correspond to a provable sequent $A \vdash B$

$$\&\,\dfrac{\vdash A, \Gamma \qquad \vdash B, \Gamma}{\vdash A \& B, \Gamma} \quad\longrightarrow\quad \mathrm{d}{\downarrow}\,\dfrac{(A \parr C) \;\&\; (B \parr C)}{(A \& B) \parr \mathrm{c}{\downarrow}\,\dfrac{C \oplus C}{C}}$$

Open deduction : symmetry

$$\text{cut}\,\frac{\vdash A, \Gamma \qquad\qquad \vdash A^\perp, \Delta}{\vdash \Gamma, \Delta} \quad\longrightarrow\quad \mathsf{s}\frac{(A \parr C) \otimes (A^\perp \parr D)}{\left(\mathsf{s}\dfrac{(A \parr C) \otimes A^\perp}{\left(\mathsf{i}{\uparrow}\dfrac{A \otimes A^\perp}{\bot}\right) \parr C}\right) \parr D}$$

If $\begin{array}{c} A \\ \| \\ B \end{array}$ in S then $\begin{array}{c} B^\perp \\ \| \\ A^\perp \end{array}$ in $S \cup \{\mathsf{i}{\downarrow}, \mathsf{i}{\uparrow}, \mathsf{s}, =\}$

A set S of inference rules is **symmetric** if

for every $\rho{\downarrow}\dfrac{A}{B} \in S$, $\rho{\uparrow}\dfrac{B^\perp}{A^\perp} \in S$

In a symmetric system, derivations can be negated

$$\begin{array}{c} A \\ \textcircled{H}\,\Big\| \\ B \end{array} \qquad\qquad =\frac{B^\perp}{\mathsf{s}\dfrac{B^\perp \otimes \mathsf{i}{\downarrow}\dfrac{1}{\begin{array}{c} A \\ \textcircled{H}\| \\ B \end{array} \parr A^\perp}}{=\dfrac{\left(\mathsf{i}{\uparrow}\dfrac{B^\perp \otimes B}{\bot}\right) \parr A^\perp}{A^\perp}}}$$

Open deduction : symmetry

$$\Phi = \boxed{\frac{E}{\frac{A}{B} \wedge \frac{C}{D}} \vee \frac{\frac{F \vee G}{H} \wedge J}{K}} \qquad \Phi^{\perp} = \boxed{\frac{\frac{B^{\perp}}{A^{\perp}} \vee \frac{D^{\perp}}{C^{\perp}}}{E^{\perp}} \wedge \frac{K^{\perp}}{\frac{H^{\perp}}{F^{\perp} \wedge G^{\perp}} \vee J^{\perp}}}$$

A set S of inference rules is **symmetric** if

for every $\rho{\downarrow}\frac{A}{B} \in S$, $\rho{\uparrow}\frac{B^{\perp}}{A^{\perp}} \in S$

In a symmetric system, derivations can be negated

Open deduction : atomicity

$$\text{cut}\,\dfrac{\vdash A, \Gamma \qquad\qquad \vdash A^\perp, \Delta}{\vdash \Gamma, \Delta} \qquad \longrightarrow \qquad \text{s}\,\dfrac{(A \parr C) \otimes (A^\perp \parr D)}{\text{s}\,\dfrac{(A \parr C) \otimes A^\perp}{\text{i}{\uparrow}\,\dfrac{(A \otimes A^\perp)}{\bot} \;\parr\; C} \;\parr\; D}$$

The cut rule can be made **atomic**, which is not possible in the sequent calculus

$$\text{c}{\downarrow}\,\dfrac{\text{d}{\uparrow}\,\dfrac{(D \mathbin{\&} E) \otimes (D^\perp \oplus E^\perp)}{\text{i}{\uparrow}\,\dfrac{D \otimes D^\perp}{\bot} \;\oplus\; \text{i}{\uparrow}\,\dfrac{E \otimes E^\perp}{\bot}}}{\bot}$$

$$\text{i}{\uparrow}\,\dfrac{=\,\dfrac{(D \otimes E) \otimes (D^\perp \parr E^\perp)}{D \;\otimes\; =\,\dfrac{\text{s}\,\dfrac{E \otimes (D^\perp \parr E^\perp)}{D^\perp \;\parr\; \text{i}{\uparrow}\,\dfrac{E \otimes E^\perp}{\bot}}}{D^\perp}}}{\bot}$$

Open deduction : atomicity

Inference rules of system SMALSg for multiplicative additive linear logic

identity

$$\mathsf{ai}{\downarrow}\,\frac{1}{a \parr a^{\perp}}$$

cut

$$\mathsf{ai}{\uparrow}\,\frac{a \otimes a^{\perp}}{\perp}$$

switch

$$\mathsf{s}\,\frac{A \otimes (B \parr C)}{(A \otimes B) \parr C}$$

$$=\frac{A}{B} \quad \text{for} \quad A = B$$

such that $\otimes$, $\parr$, $\&$, $\oplus$ are commutative and associative and have units 1, $\perp$, $\top$, 0

contraction

$$\mathsf{c}{\downarrow}\,\frac{A \oplus A}{A}$$

cocontraction

$$\mathsf{c}{\uparrow}\,\frac{A}{A \mathbin{\&} A}$$

$$\mathsf{d}{\downarrow}\,\frac{(A \parr B) \mathbin{\&} (C \parr D)}{(A \mathbin{\&} C) \parr (B \oplus D)}$$

$$\mathsf{t}{\downarrow}\,\frac{0}{A}$$

$$\mathsf{t}{\uparrow}\,\frac{A}{\top}$$

$$\mathsf{d}{\uparrow}\,\frac{(A \mathbin{\&} B) \otimes (C \oplus D)}{(A \otimes C) \oplus (B \otimes D)}$$

Note that there is a difference between our cocontraction and DiLL's cocontraction: theirs is a flipped contraction, ours is a flipped *and dualised* contraction.

Open deduction : atomicity

Inference rules of system SKSg for propositional classical logic

identity

$$\text{ai}\!\downarrow \frac{\top}{a \vee a^{\perp}}$$

cut

$$\text{ai}\!\uparrow \frac{a \wedge a^{\perp}}{\perp}$$

switch

$$\text{s}\,\frac{A \wedge (B \vee C)}{(A \wedge B) \vee C}$$

$$=\frac{A}{B}$$ for $A = B$

such that $\wedge, \vee$ are commutative and associative and have units $\top, \perp$,

contraction

$$\text{c}\!\downarrow \frac{A \vee A}{A}$$

cocontraction

$$\text{c}\!\uparrow \frac{A}{A \wedge A}$$

weakening

$$\text{w}\!\downarrow \frac{\perp}{A}$$

coweakening

$$\text{w}\!\uparrow \frac{A}{\top}$$

Open deduction : atomicity

Inference rules of system SKSg for propositional classical logic

identity

$$\mathsf{ai}\!\downarrow \frac{\top}{a \vee a^{\perp}}$$

cut

$$\mathsf{ai}\!\uparrow \frac{a \wedge a^{\perp}}{\perp}$$

switch

$$\mathsf{s}\, \frac{A \wedge (B \vee C)}{(A \wedge B) \vee C}$$

$$= \frac{A}{B}$$

contraction

$$\mathsf{c}\!\downarrow \frac{A \vee A}{A}$$

cocontraction

$$\mathsf{c}\!\uparrow \frac{A}{A \wedge A}$$

weakening

$$\mathsf{w}\!\downarrow \frac{\perp}{A}$$

coweakening

$$\mathsf{w}\!\uparrow \frac{A}{\top}$$

$$\mathsf{c}\!\downarrow \frac{\left(= \dfrac{A}{A \vee \mathsf{w}\!\downarrow \dfrac{\perp}{C}} \wedge = \dfrac{B}{B \vee \mathsf{w}\!\downarrow \dfrac{\perp}{D}} \right) \vee \left(= \dfrac{C}{\mathsf{w}\!\downarrow \dfrac{\perp}{A} \vee C} \wedge = \dfrac{D}{\mathsf{w}\!\downarrow \dfrac{\perp}{B} \vee D} \right)}{(A \vee C) \wedge (B \vee D)}$$

Open deduction : atomicity

Inference rules of system SKSg for propositional classical logic

identity

$$\mathsf{ai}{\downarrow}\,\frac{\top}{a \vee a^{\perp}}$$

cut

$$\mathsf{ai}{\uparrow}\,\frac{a \wedge a^{\perp}}{\perp}$$

switch

$$\mathsf{s}\,\frac{A \wedge (B \vee C)}{(A \wedge B) \vee C}$$

$$=\frac{A}{B}$$

contraction

$$\mathsf{c}{\downarrow}\,\frac{A \vee A}{A}$$

cocontraction

$$\mathsf{c}{\uparrow}\,\frac{A}{A \wedge A}$$

weakening

$$\mathsf{w}{\downarrow}\,\frac{\perp}{A}$$

coweakening

$$\mathsf{w}{\uparrow}\,\frac{A}{\top}$$

$$\mathsf{m}\,\frac{(A \wedge B) \vee (C \wedge D)}{(A \vee C) \wedge (B \vee D)}$$

medial

Open deduction : atomicity

Inference rules of system SKSg for propositional classical logic

identity

$$\text{ai}\downarrow\frac{\top}{a \vee a^{\perp}}$$

cut

$$\text{ai}\uparrow\frac{a \wedge a^{\perp}}{\bot}$$

switch

$$\text{s}\frac{A \wedge (B \vee C)}{(A \wedge B) \vee C}$$

$$=\frac{A}{B}$$

contraction

$$\text{c}\downarrow\frac{A \vee A}{A}$$

cocontraction

$$\text{c}\uparrow\frac{A}{A \wedge A}$$

weakening

$$\text{w}\downarrow\frac{\bot}{A}$$

coweakening

$$\text{w}\uparrow\frac{A}{\top}$$

$$\text{m}\frac{(A \wedge B) \vee (A \wedge B)}{\text{c}\downarrow\dfrac{(A \vee A)}{A} \wedge \text{c}\downarrow\dfrac{(B \vee B)}{B}}$$

medial allows to decompose contraction to atomic form

Open deduction : atomicity

Inference rules of system SKSg for propositional classical logic

identity

$$ai{\downarrow}\,\frac{\top}{a \vee a^{\perp}}$$

cut

$$ai{\uparrow}\,\frac{a \wedge a^{\perp}}{\bot}$$

switch

$$s\,\frac{A \wedge (B \vee C)}{(A \wedge B) \vee C}$$

$$=\frac{A}{B}$$

contraction

$$ac{\downarrow}\,\frac{a \vee a}{a}$$

cocontraction

$$ac{\uparrow}\,\frac{a}{a \wedge a}$$

medial

$$m\,\frac{(A \wedge B) \vee (C \wedge D)}{(A \vee C) \wedge (B \vee D)}$$

$$uc{\downarrow}\,\frac{\top \vee \top}{\top}$$

$$uc{\uparrow}\,\frac{\bot}{\bot \wedge \bot}$$

weakening

$$w{\downarrow}\,\frac{\bot}{A}$$

coweakening

$$w{\uparrow}\,\frac{A}{\top}$$

medial allows to decompose contraction to atomic form

Open deduction : atomicity

Inference rules of system SKS for propositional classical logic

identity

$$\mathsf{ai}{\downarrow}\,\frac{\top}{a \vee a^{\perp}}$$

cut

$$\mathsf{ai}{\uparrow}\,\frac{a \wedge a^{\perp}}{\perp}$$

switch

$$\mathsf{s}\,\frac{A \wedge (B \vee C)}{(A \wedge B) \vee C}$$

$$=\frac{A}{B}$$

contraction

$$\mathsf{ac}{\downarrow}\,\frac{a \vee a}{a}$$

cocontraction

$$\mathsf{ac}{\uparrow}\,\frac{a}{a \wedge a}$$

medial

$$\mathsf{m}\,\frac{(A \wedge B) \vee (C \wedge D)}{(A \vee C) \wedge (B \vee D)}$$

$$\mathsf{uc}{\downarrow}\,\frac{\top \vee \top}{\top}$$

$$\mathsf{uc}{\uparrow}\,\frac{\perp}{\perp \wedge \perp}$$

weakening

$$\mathsf{w}{\downarrow}\,\frac{\perp}{a}$$

coweakening

$$\mathsf{w}{\uparrow}\,\frac{a}{\top}$$

Every structural rule is **atomic** so every inference step is local

Open deduction : bureaucracy

$$\text{weakR}\,\dfrac{\text{contR}\,\dfrac{\vdash A, A, \Gamma}{\vdash A, \Gamma}}{\vdash A, B, \Gamma} \quad = \quad \text{contR}\,\dfrac{\text{weakR}\,\dfrac{\vdash A, A, \Gamma}{\vdash A, A, B, \Gamma}}{\vdash A, B, \Gamma}$$

$$\searrow \qquad\qquad \swarrow$$

$$= \dfrac{A \vee A \vee C}{c\!\downarrow\dfrac{A \vee A}{A} \;\vee\; w\!\downarrow\dfrac{\bot}{B} \;\vee\; C}$$

In open deduction we can reduce the bureaucracy of parallel independent sub-derivations (similarly to proof nets)

Open deduction : bureaucracy

$$\text{cut}\,\dfrac{\vdash A, \Gamma \qquad \text{contR}\,\dfrac{\vdash A^{\perp}, B, B, \Delta}{\vdash A^{\perp}, B, \Delta}}{\vdash \Gamma, B, \Delta}$$

$$\downarrow$$

$$2s\,\dfrac{(A \vee C) \wedge \left(A^{\perp} \vee c{\downarrow}\dfrac{B \vee B}{B} \vee D\right)}{i{\uparrow}\dfrac{A \wedge A^{\perp}}{\perp} \vee C \vee B \vee D}$$

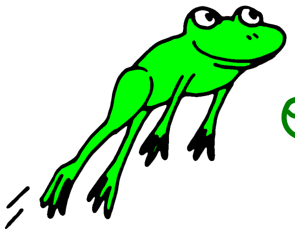
$$\text{cut}\,\dfrac{\vdash A, \Gamma \qquad \vdash A^{\perp}, B, B, \Delta}{\text{contR}\,\dfrac{\vdash \Gamma, B, B, \Delta}{\vdash \Gamma, B, \Delta}}$$

$$\downarrow$$

$$2s\,\dfrac{(A \vee C) \wedge (A^{\perp} \vee B \vee B \vee D)}{i{\uparrow}\dfrac{A \wedge A^{\perp}}{\perp} \vee C \vee c{\downarrow}\dfrac{B \vee B}{B} \vee D}$$

$$\downarrow \qquad \searrow$$

$$\left\langle c{\downarrow}\dfrac{B \vee B}{B} \middle/ X \right\rangle \; 2s\,\dfrac{(A \vee C) \wedge (A^{\perp} \vee X \vee D)}{i{\uparrow}\dfrac{A \wedge A^{\perp}}{\perp} \vee C \vee X \vee D}$$



We can develop
explicit substitutions
for derivations

Open deduction

Derivations are symmetric

Inference rules are atomic or linear

Derivations are low in bureaucracy

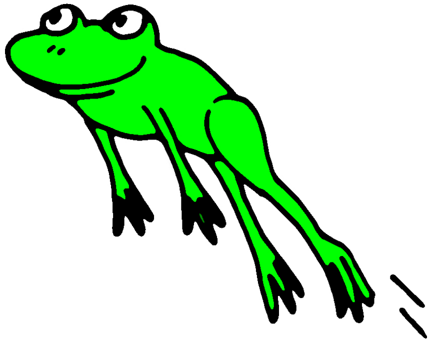
We can see the geometric nature of derivations clearly and develop normalisation procedures.

Summary

We introduced open deduction and saw how symmetry, atomicity, and low bureaucracy lead to good normalisation (for propositional classical logic)

Tomorrow : Normalisation for linear logics

Proof compression mechanisms



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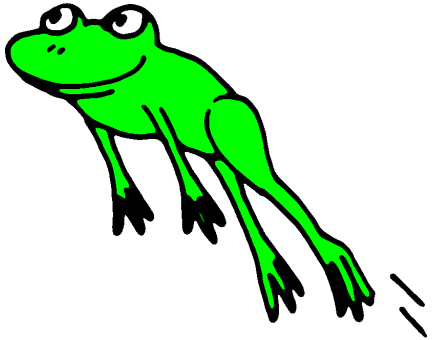

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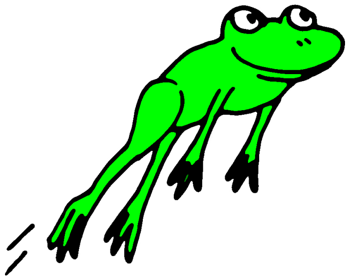
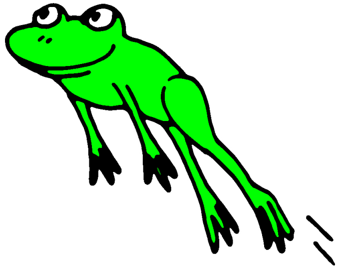


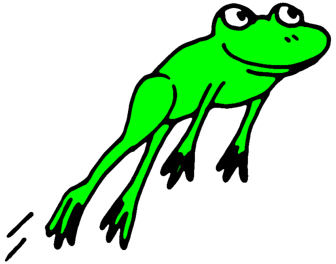
Victoria Barrett
Inria Saclay, Équipe Partout



Proof Society 2026, Aussois

Today : Normalisation via atomic flows for propositional classical logic

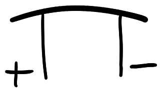
Normalisation via splitting and decomposition

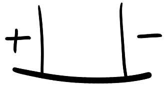


Atomic Flows

From derivations to flows

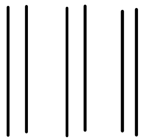
$$\mathsf{ai}{\downarrow}\,\frac{\top}{a \vee a^{\perp}}$$





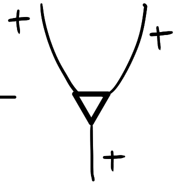
$$\mathsf{ai}{\uparrow}\,\frac{a \wedge a^{\perp}}{\perp}$$

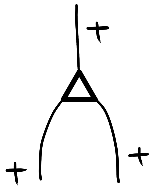
$$\mathsf{s}\,\frac{A \wedge (B \vee C)}{(A \wedge B) \vee C}$$



$$\mathsf{uc}{\downarrow}\,\frac{\top \vee \top}{\top}$$

$$\mathsf{ac}{\downarrow}\,\frac{a \vee a}{a}$$



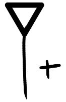


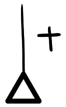
$$\mathsf{ac}{\uparrow}\,\frac{a}{a \wedge a}$$

$$\mathsf{m}\,\frac{(A \wedge B) \vee (C \wedge D)}{(A \vee C) \wedge (B \vee D)}$$

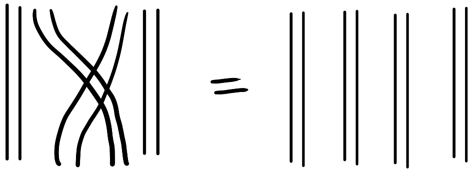
$$\mathsf{uc}{\uparrow}\,\frac{\perp}{\perp \wedge \perp}$$

$$\mathsf{w}{\downarrow}\,\frac{\perp}{a}$$

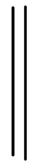




$$\mathsf{w}{\uparrow}\,\frac{a}{\top}$$

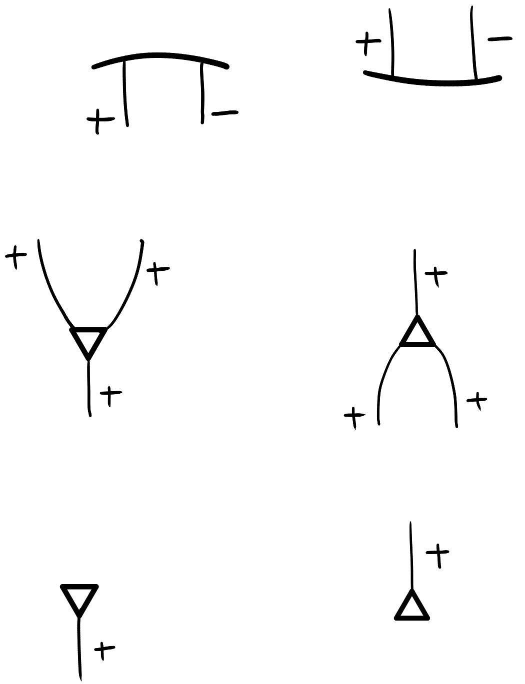


$$=\,\frac{A}{B}$$



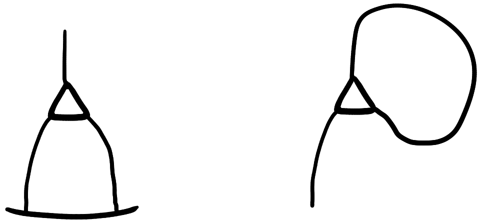
We associate a generator to each structural rule

From derivations to flows



Atomic flows are graphs whose nodes are labelled by generators and whose edges are polarised and vertically directed

Forbidden :

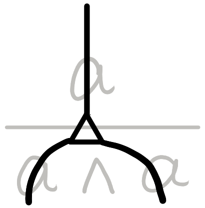


We associate a generator to each structural rule

From derivations to flows

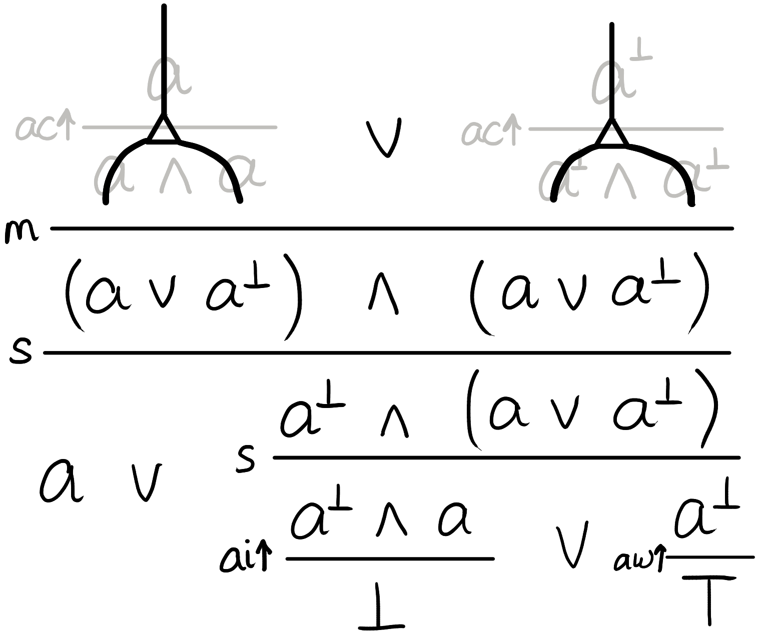
$$
\mathsf{m}\,\dfrac{\mathsf{ac}\!\uparrow\dfrac{a}{a\wedge a} \quad\vee\quad \mathsf{ac}\!\uparrow\dfrac{a^{\perp}}{a^{\perp}\wedge a^{\perp}}}{\mathsf{s}\,\dfrac{(a\vee a^{\perp}) \;\wedge\; (a\vee a^{\perp})}{a \;\vee\; \mathsf{s}\,\dfrac{a^{\perp}\wedge(a\vee a^{\perp})}{\mathsf{ai}\!\uparrow\dfrac{a^{\perp}\wedge a}{\perp} \;\vee\; \mathsf{aw}\!\uparrow\dfrac{a^{\perp}}{\top}}}}
$$

From derivations to flows



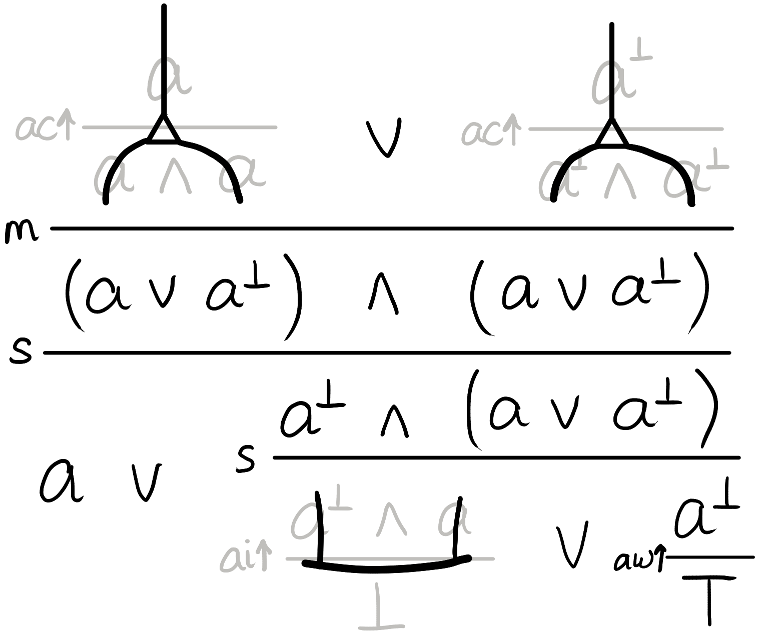
$$\mathsf{m}\,\dfrac{\mathsf{ac}{\uparrow}\dfrac{a}{a \wedge a} \quad \vee \quad \mathsf{ac}{\uparrow}\dfrac{a^\perp}{a^\perp \wedge a^\perp}}{\mathsf{s}\,\dfrac{(a \vee a^\perp) \quad \wedge \quad (a \vee a^\perp)}{a \quad \vee \quad \mathsf{s}\,\dfrac{a^\perp \wedge (a \vee a^\perp)}{\mathsf{ai}{\uparrow}\dfrac{a^\perp \wedge a}{\bot} \quad \vee \quad \mathsf{aw}{\uparrow}\dfrac{a^\perp}{\top}}}}$$

From derivations to flows

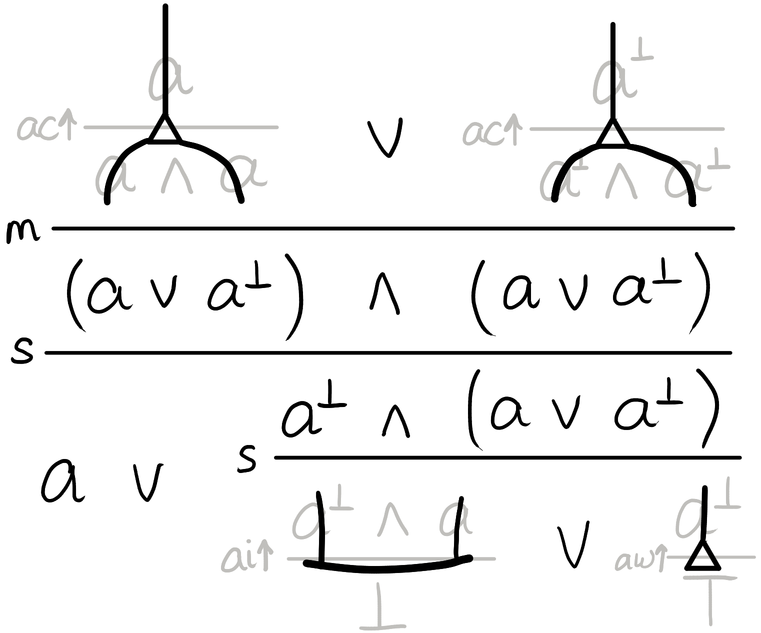


$$\dfrac{\mathsf{ac}{\uparrow}\,\dfrac{a}{a \wedge a} \;\vee\; \mathsf{ac}{\uparrow}\,\dfrac{a^{\perp}}{a^{\perp} \wedge a^{\perp}}}{\mathsf{m}\;\dfrac{(a \vee a^{\perp}) \wedge (a \vee a^{\perp})}{\mathsf{s}\;\; a \vee \mathsf{s}\,\dfrac{a^{\perp} \wedge (a \vee a^{\perp})}{\mathsf{ai}{\uparrow}\,\dfrac{a^{\perp} \wedge a}{\perp} \;\vee\; \mathsf{aw}{\uparrow}\,\dfrac{a^{\perp}}{\top}}}}$$

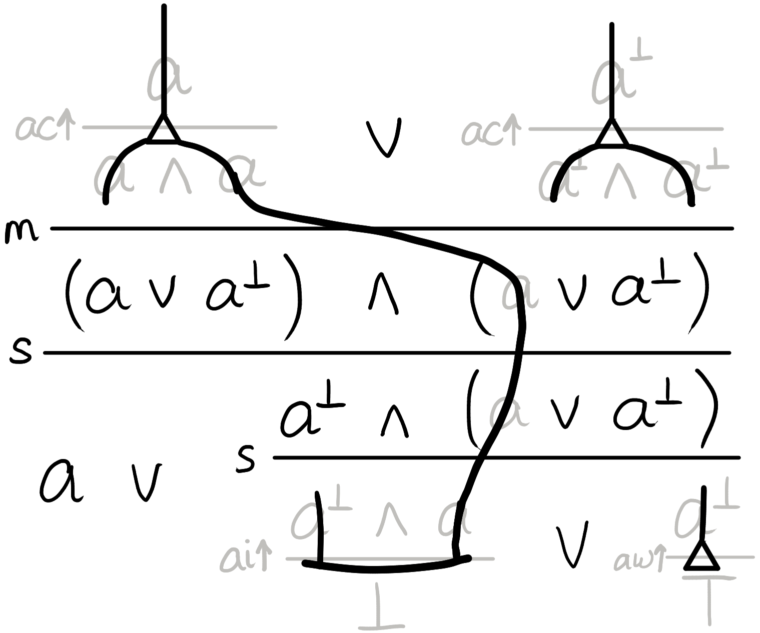
From derivations to flows



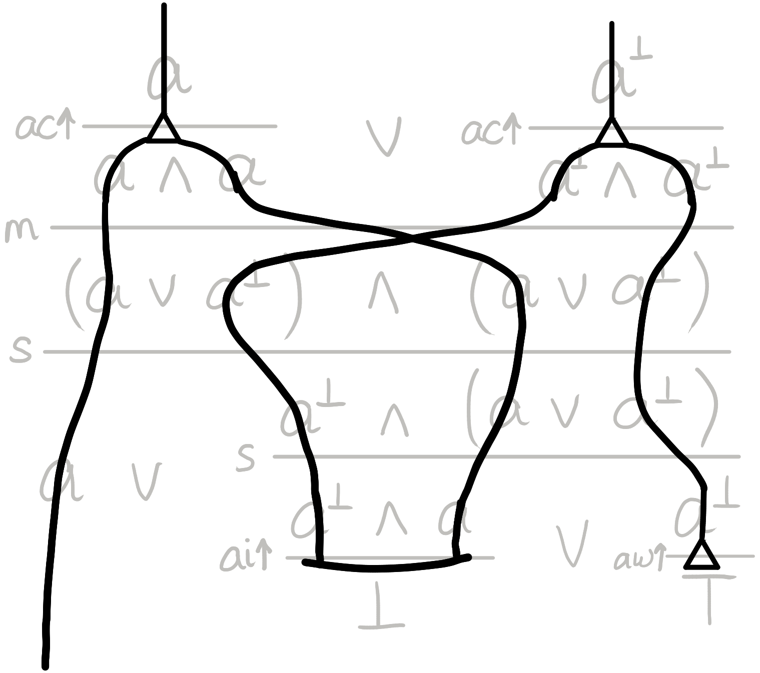
From derivations to flows



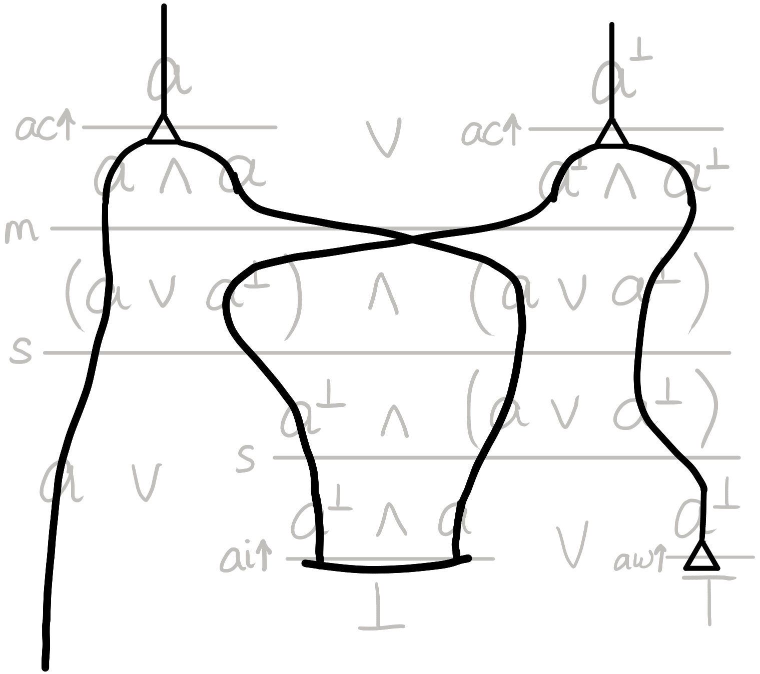
From derivations to flows



From derivations to flows

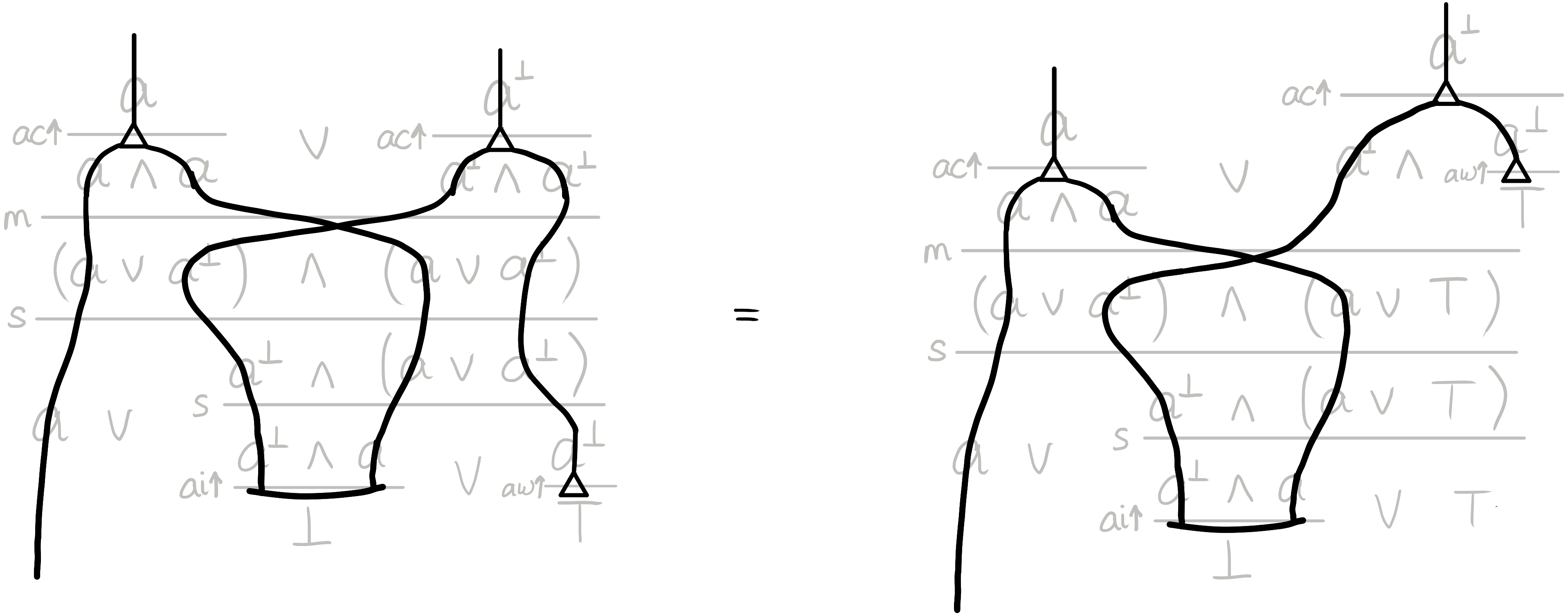


From derivations to flows



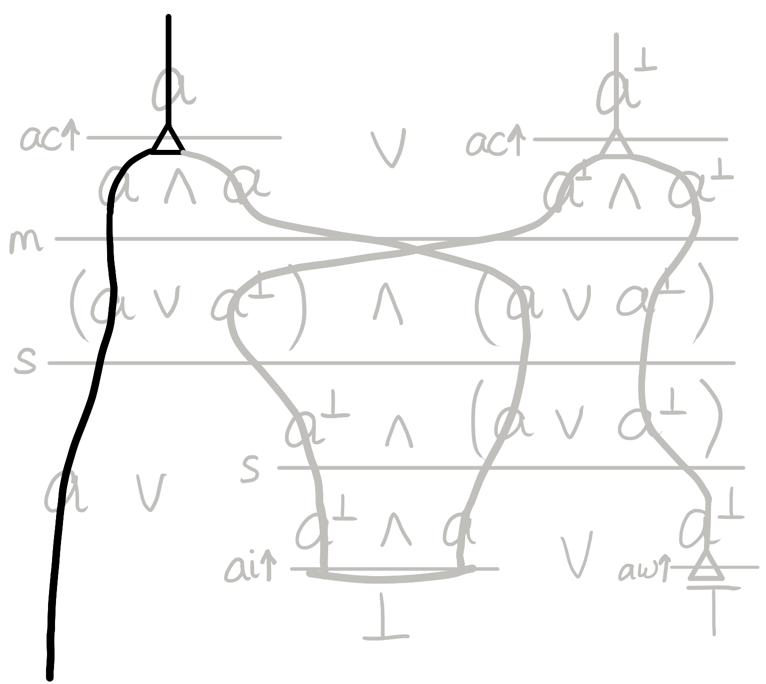
$$\mathsf{m}\,\frac{\mathsf{ac}\uparrow\dfrac{a}{a\wedge a} \;\vee\; \mathsf{ac}\uparrow\dfrac{a^\perp}{a^\perp \wedge \mathsf{aw}\uparrow\dfrac{a^\perp}{\top}}}{\mathsf{s}\,\dfrac{(a\vee a^\perp)\wedge(a\vee\top)}{a \vee \mathsf{s}\,\dfrac{a^\perp\wedge(a\vee\top)}{\mathsf{ai}\uparrow\dfrac{a^\perp\wedge a}{\bot}\vee\top}}}$$

From derivations to flows



We retain only the structural information of derivations, up to continuous deformation of the flows

From derivations to flows



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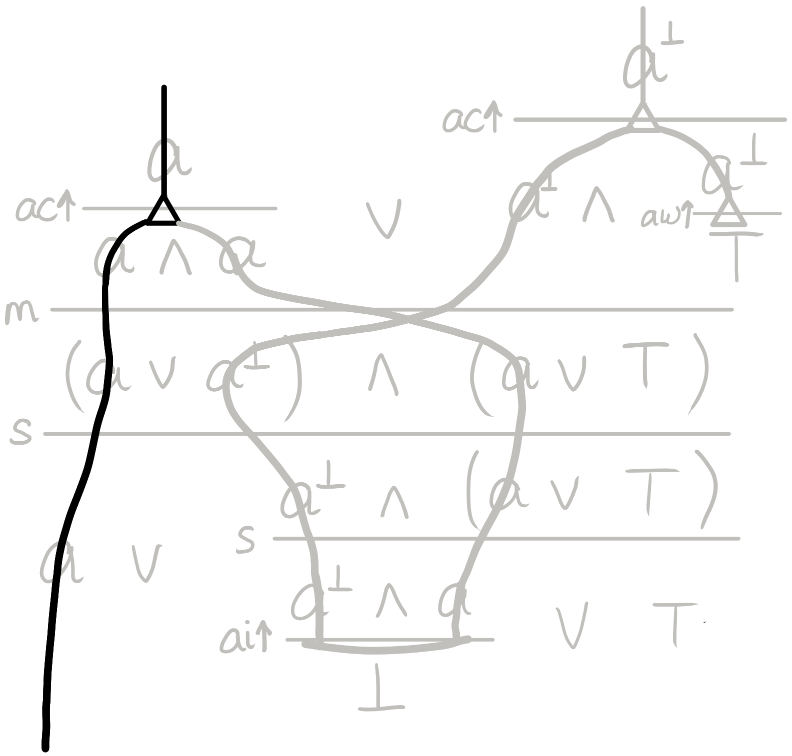
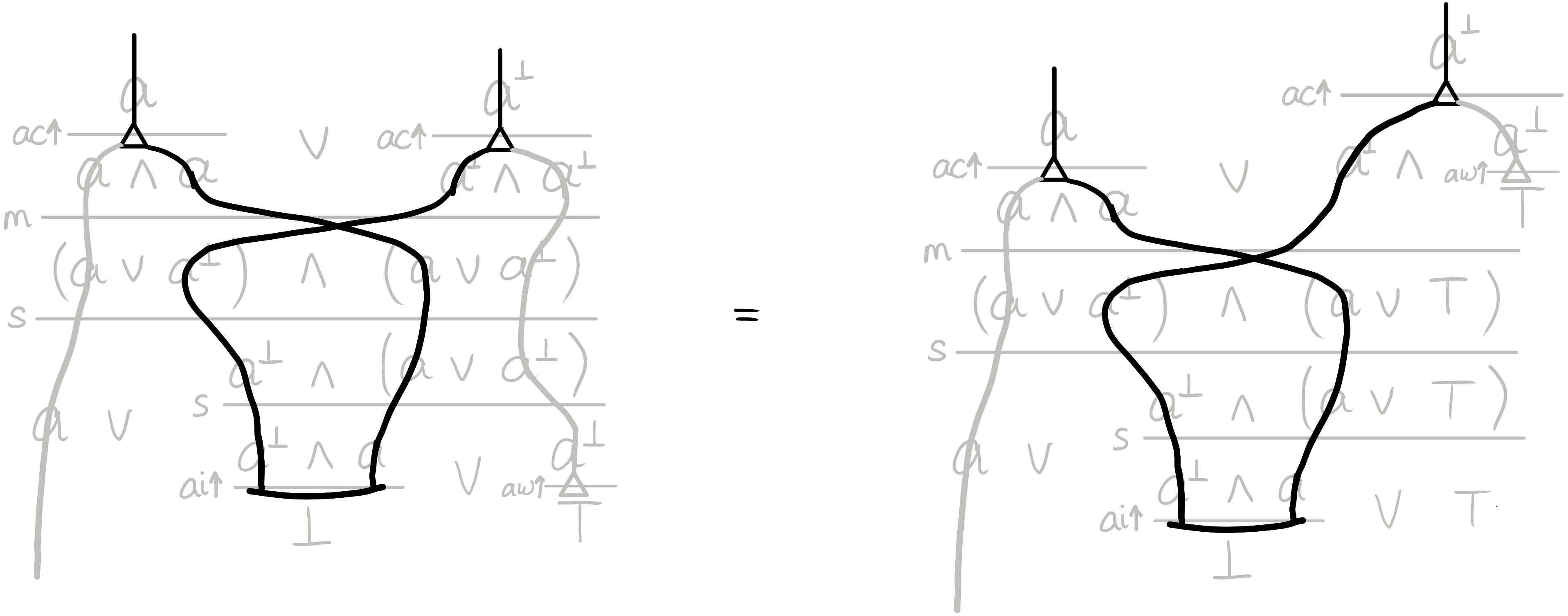


We retain only the structural information of derivations, up to continuous deformation of the flows

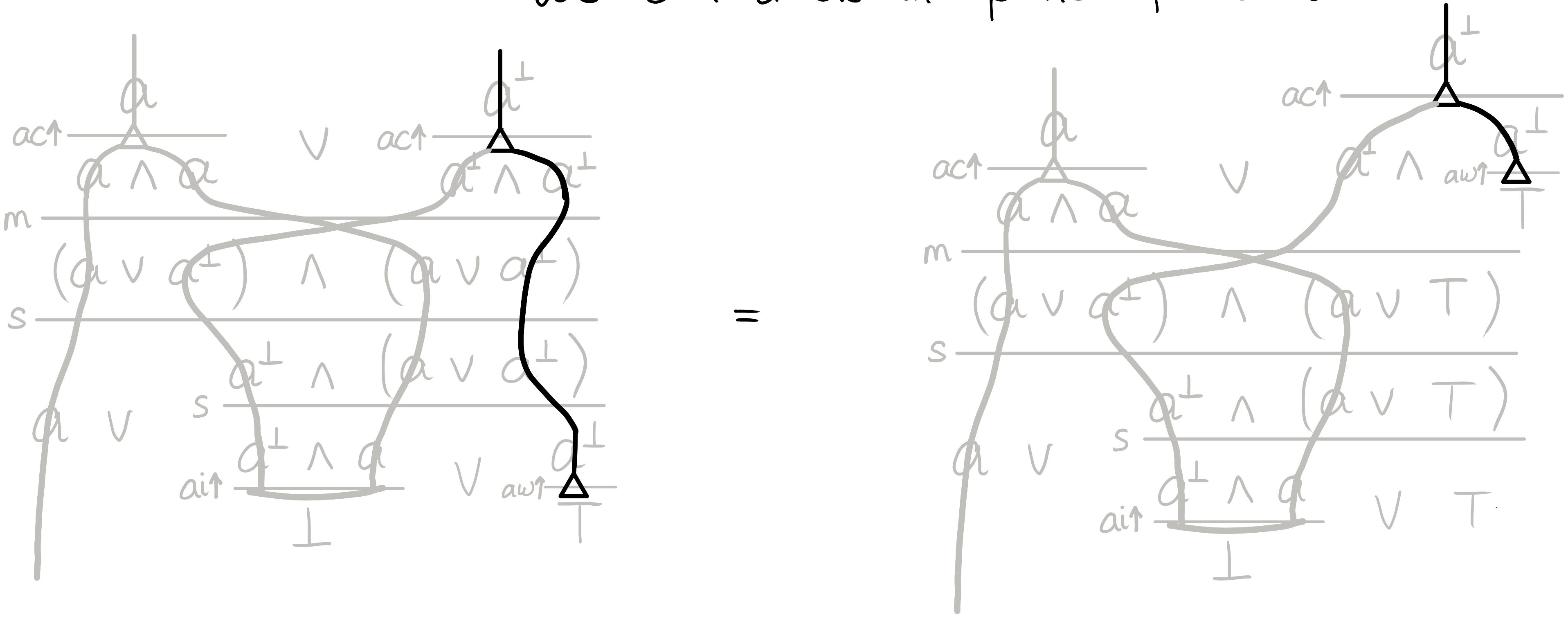
From derivations to flows



We retain only the structural information of derivations, up to continuous deformation of the flows

From derivations to flows

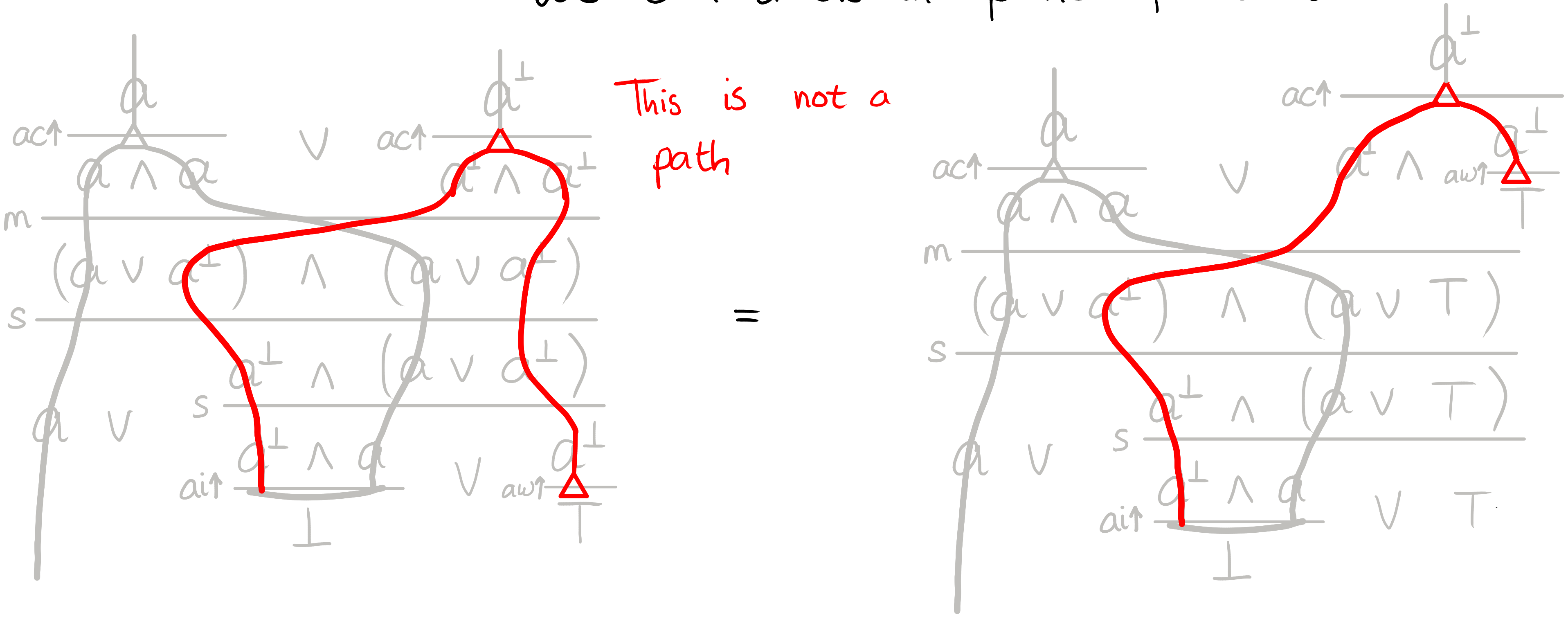
We can track the paths of atoms



We retain only the structural information of derivations, up to continuous deformation of the flows

From derivations to flows

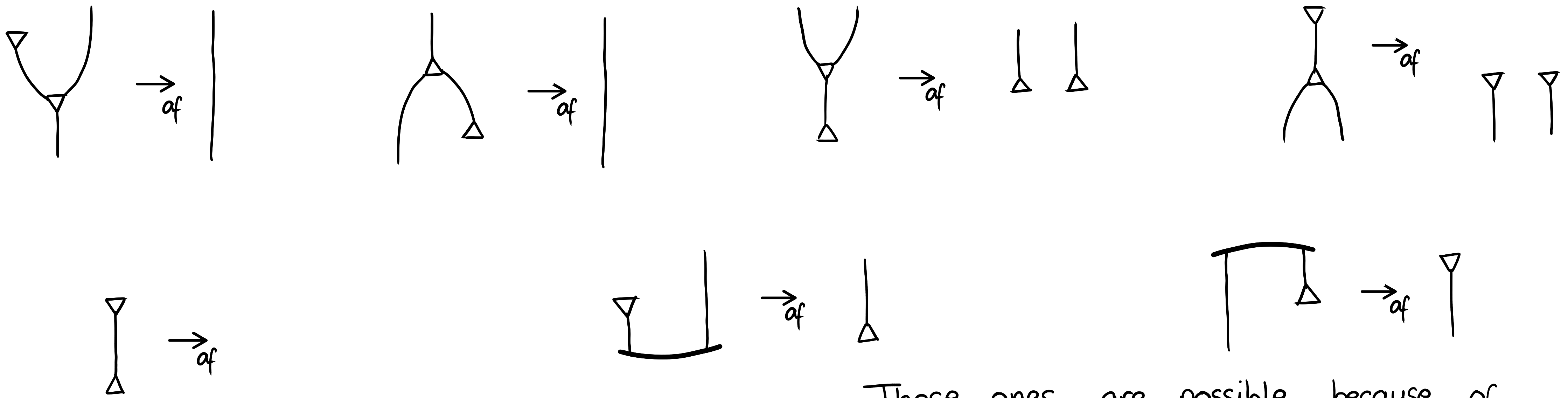
We can track the paths of atoms

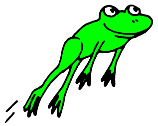


This is not a path

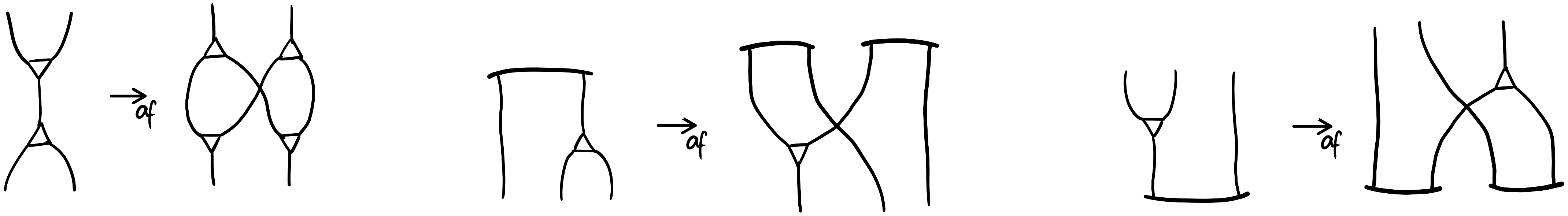
We retain only the structural information of derivations, up to continuous deformation of the flows

Rewriting flows





These ones are possible because of the symmetry of the rules



Rewriting flows lifts to derivations

$$K\left\{ \mathsf{ac}{\downarrow}\,\frac{a \vee a}{a} \right\}$$

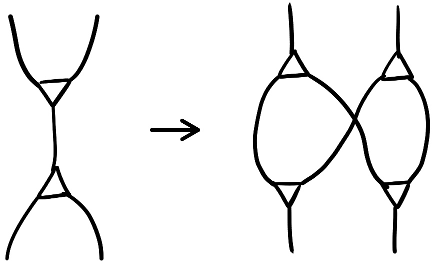
$$\Phi(a)\ \Big\|$$

$$H\left\{ \mathsf{ac}{\uparrow}\,\frac{a}{a \wedge a} \right\}$$

$$K\left\{ \mathsf{m}\,\dfrac{\mathsf{ac}{\uparrow}\,\dfrac{a}{a \wedge a} \ \vee\ \mathsf{ac}{\uparrow}\,\dfrac{a}{a \wedge a}}{\mathsf{ac}{\downarrow}\,\dfrac{a \vee a}{a} \ \wedge\ \mathsf{ac}{\downarrow}\,\dfrac{a \vee a}{a}} \right\}$$

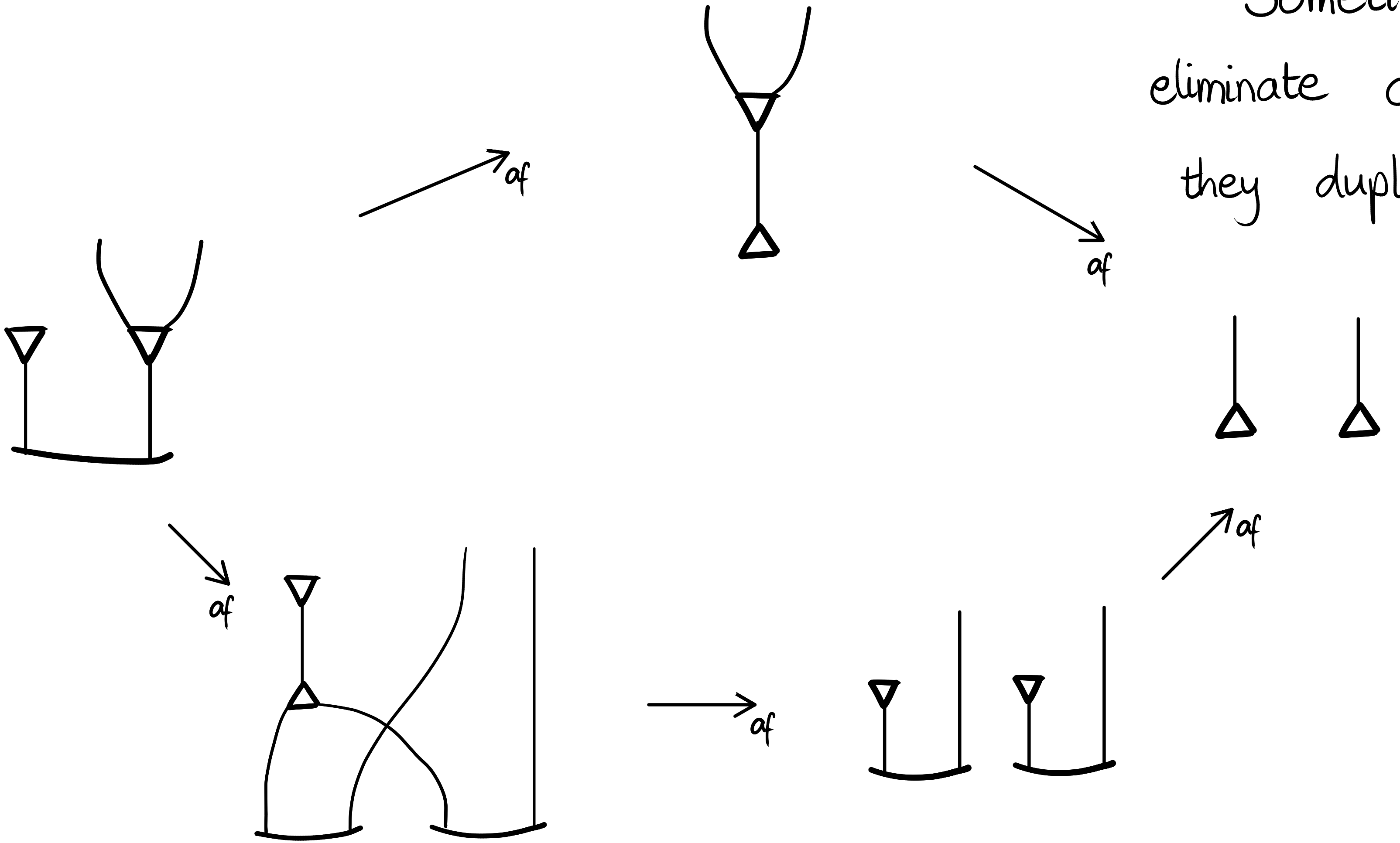
$$\Phi(a \wedge a)\ \Big\|$$

$$H\{a \wedge a\}$$



Rewriting flows is locally confluent

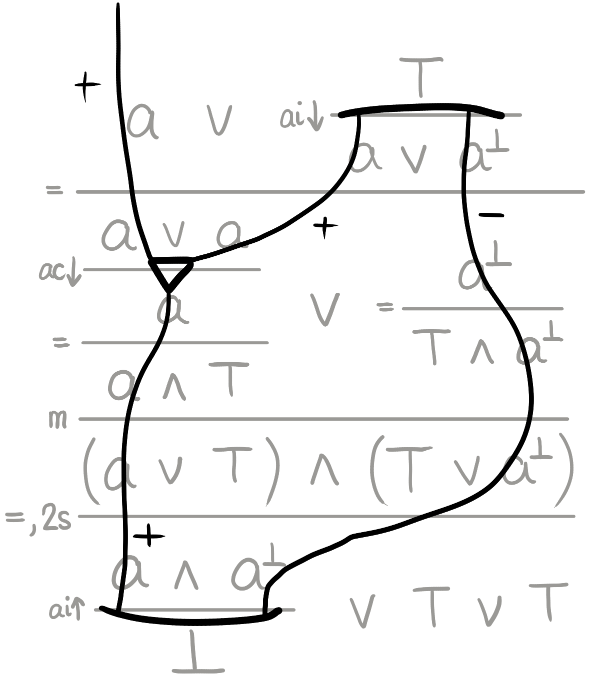
Sometimes rewriting steps eliminate cuts and sometimes they duplicate them



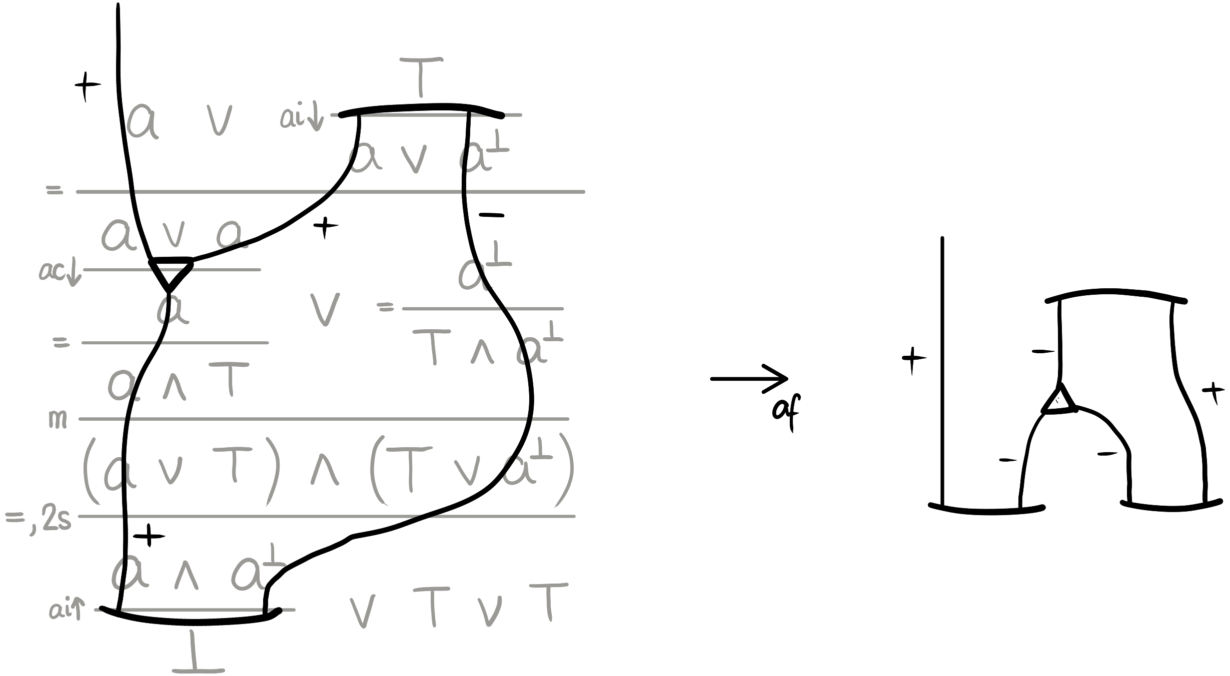
Rewriting flows doesn't always terminate

$$
{\scriptstyle =,2s}\dfrac{{\scriptstyle m}\dfrac{{\scriptstyle =}\dfrac{a \vee {\scriptstyle ai\downarrow}\dfrac{\top}{a \vee a^{\perp}}}{{\scriptstyle =}\dfrac{{\scriptstyle ac\downarrow}\dfrac{a \vee a}{a}}{a \wedge \top} \vee {\scriptstyle =}\dfrac{a^{\perp}}{\top \wedge a^{\perp}}}}{(a \vee \top) \wedge (\top \vee a^{\perp})}}{{\scriptstyle ai\uparrow}\dfrac{a \wedge a^{\perp}}{\perp} \vee \top \vee \top}
$$

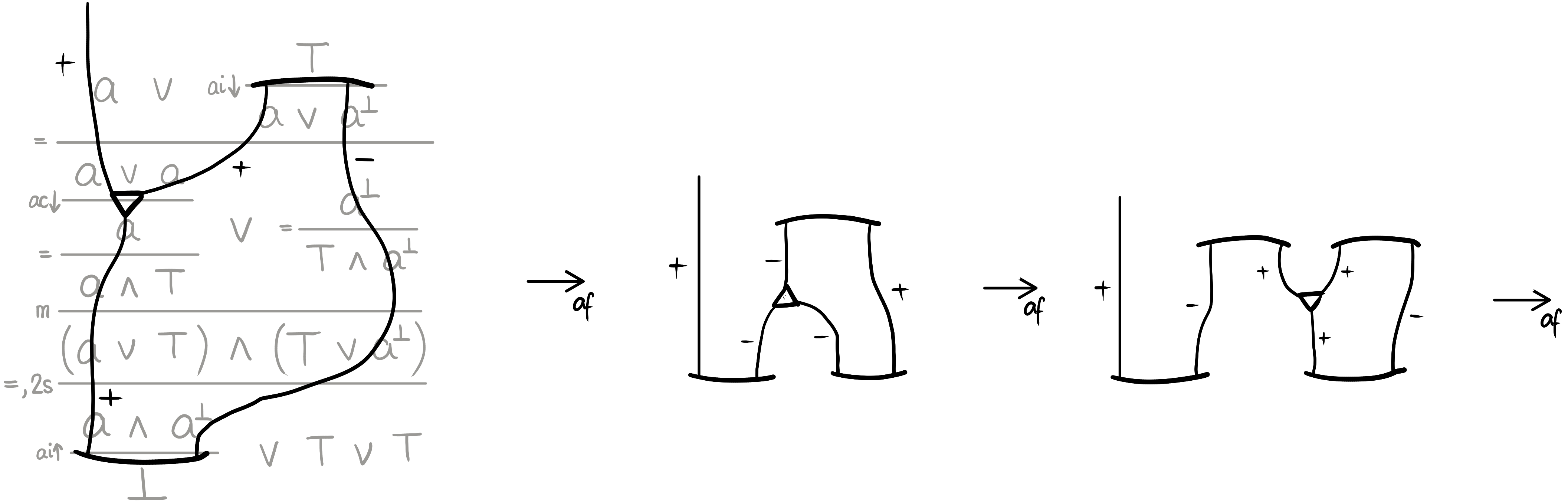
Rewriting flows doesn't always terminate



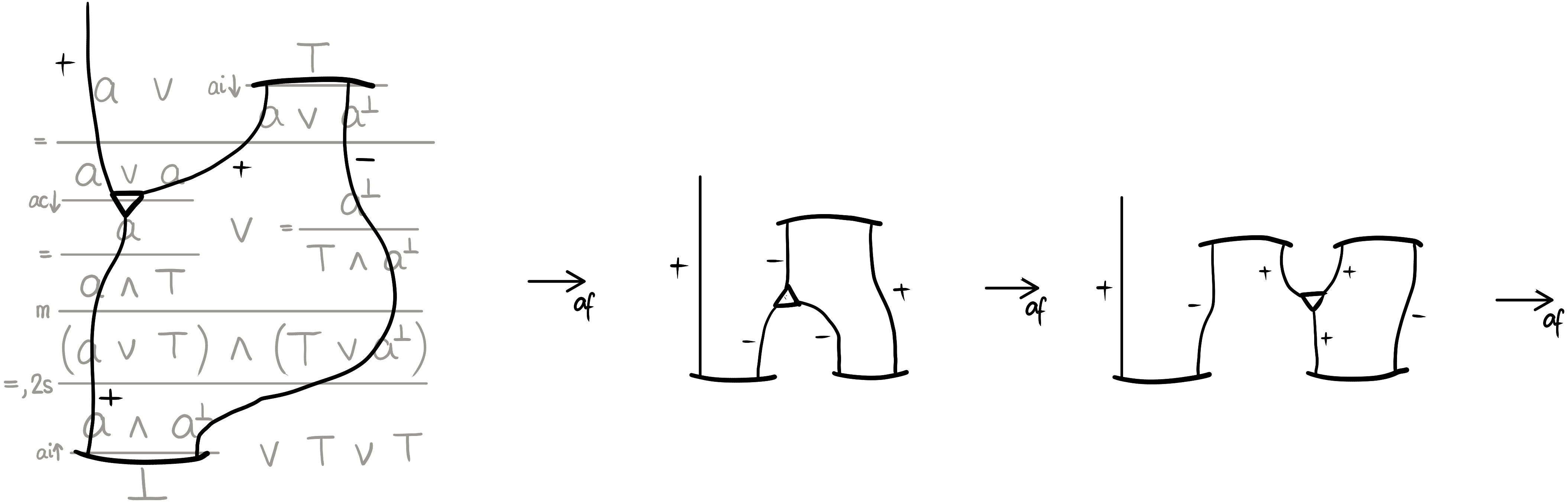
Rewriting flows doesn't always terminate



Rewriting flows doesn't always terminate

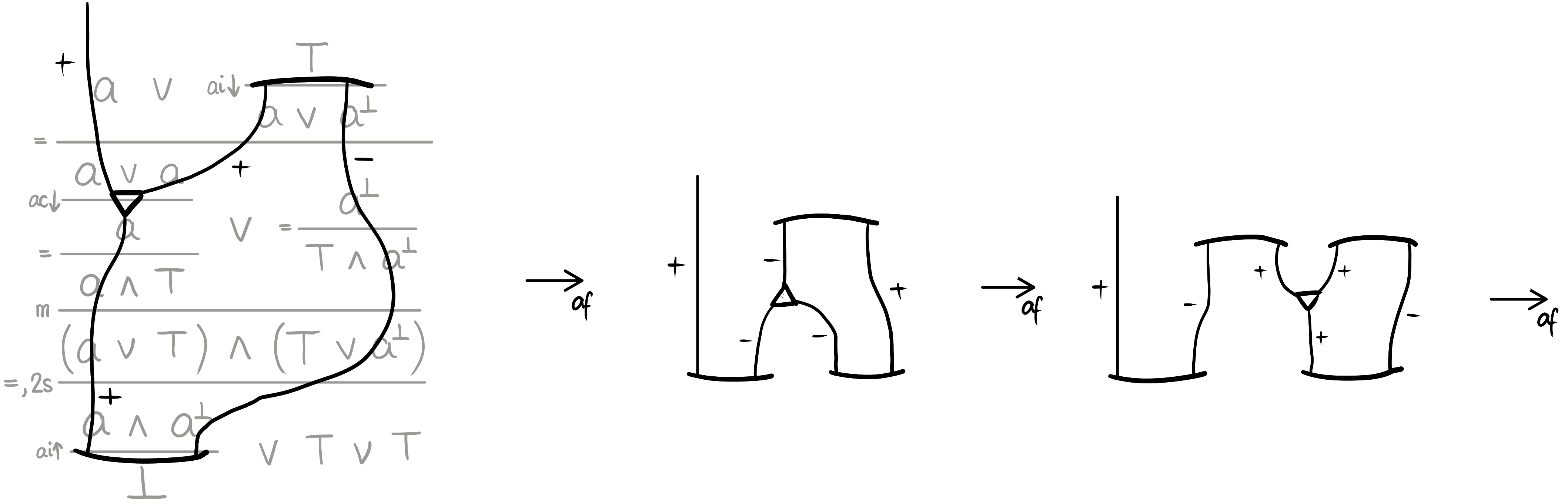


Rewriting flows doesn't always terminate



The problem arises when a (co-)contraction gets into a path between an identity and a cut

Rewriting flows doesn't always terminate



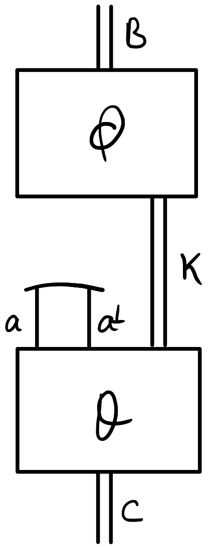
The problem arises when a (co-)contraction gets into a path between an identity and a cut

We should break the paths between identities and cuts

Breaking paths in atomic flows

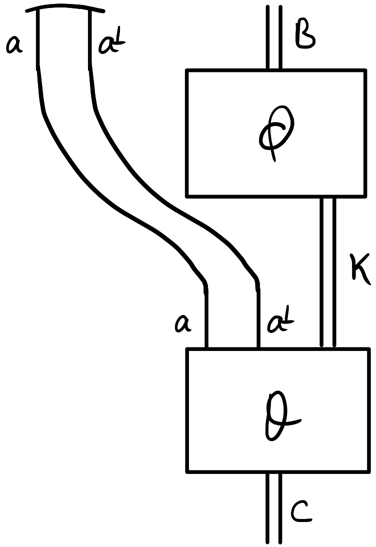
We can pull identities up derivations

$$\begin{array}{c} B \\ \Phi \Big\Vert \\ K\left\{ \mathsf{ai}\!\downarrow \dfrac{\top}{a \vee a^{\perp}} \right\} \\ \Theta \Big\Vert \\ C \end{array}$$



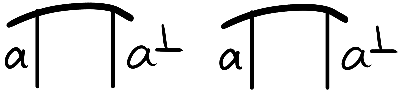
$$\longrightarrow$$

$$\begin{array}{c} \boxed{\mathsf{ai}\!\downarrow \dfrac{\top}{a \vee a^{\perp}} \;\wedge\; \begin{array}{c} B \\ \Phi \Big\Vert \\ K\{\top\} \end{array}} \\ \Big\Vert \; \mathsf{s}, \; = \\ K\{a \vee a^{\perp}\} \\ \Theta \Big\Vert \\ C \end{array}$$

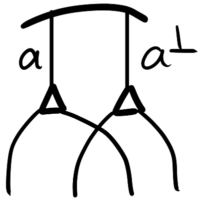


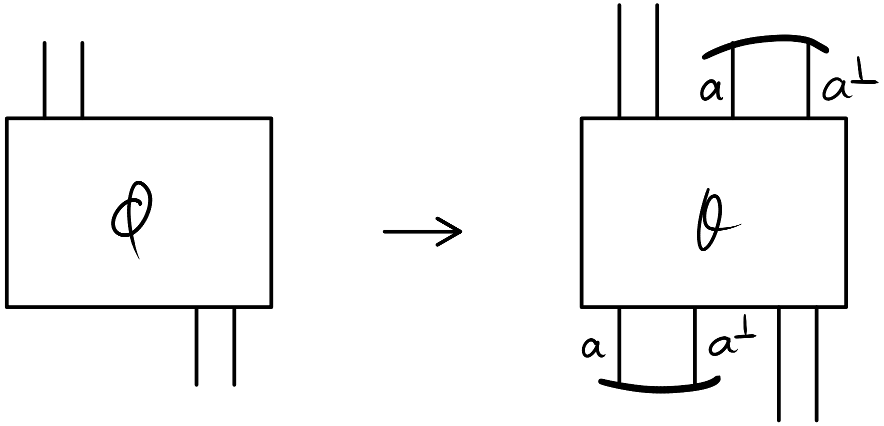
Breaking paths in atomic flows

We can gather all the cuts and identities on a single atom



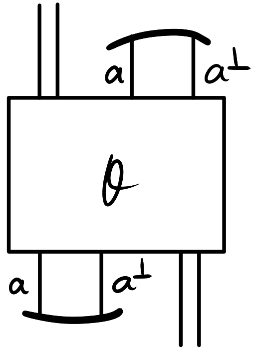
$$\mathsf{ai}{\downarrow}\,\frac{\top}{a \vee a^{\perp}} \;\wedge\; \mathsf{ai}{\downarrow}\,\frac{\top}{a \vee a^{\perp}} \;\longrightarrow\; \mathsf{m}\,\frac{\mathsf{ai}{\downarrow}\,\dfrac{\top}{\mathsf{ac}{\uparrow}\,\dfrac{a}{a \wedge a} \;\vee\; \mathsf{ac}{\uparrow}\,\dfrac{a^{\perp}}{a^{\perp} \wedge a^{\perp}}}}{(a \vee a^{\perp}) \;\wedge\; (a \vee a^{\perp})}$$



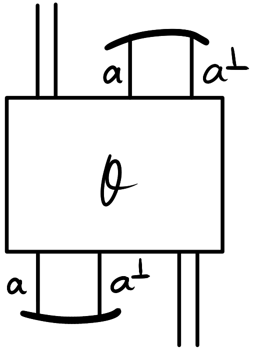


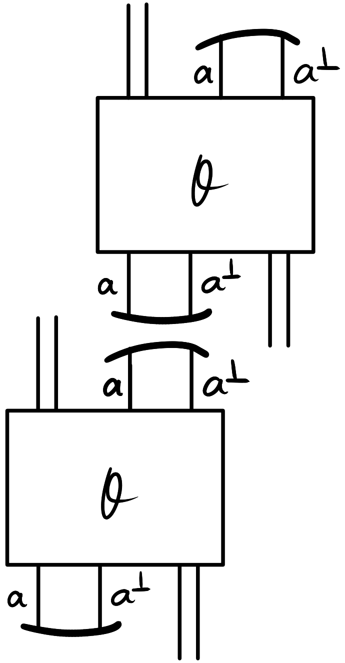
where θ contains no cuts or identities on a

Breaking paths in atomic flows

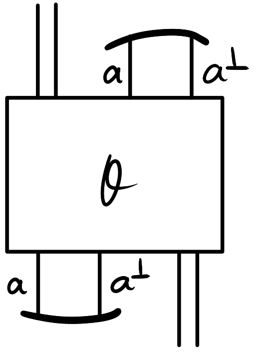


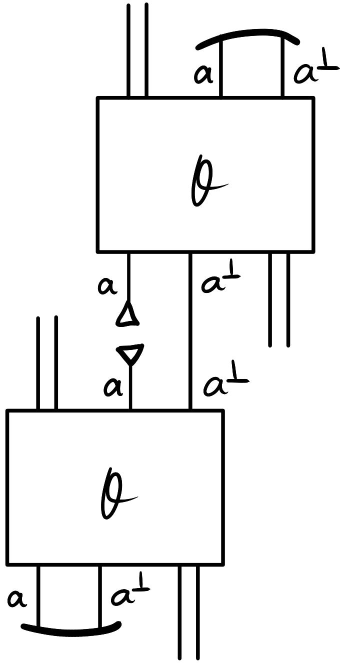
Breaking paths in atomic flows





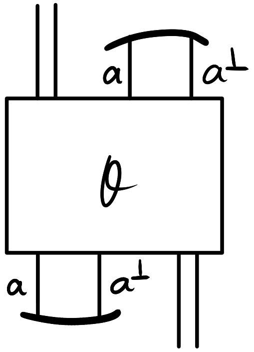
Breaking paths in atomic flows

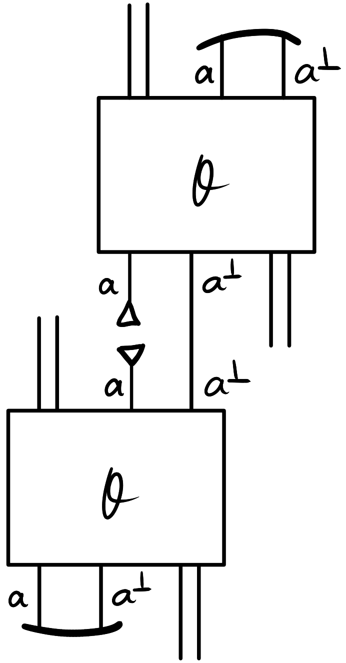




$$=\frac{\text{aw}\uparrow\dfrac{a}{\top}\;\wedge\;a^{\perp}}{\text{aw}\downarrow\dfrac{\bot}{a}\;\vee\;a^{\perp}}$$

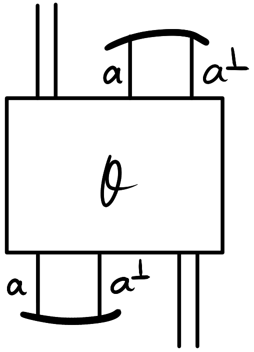
Breaking paths in atomic flows

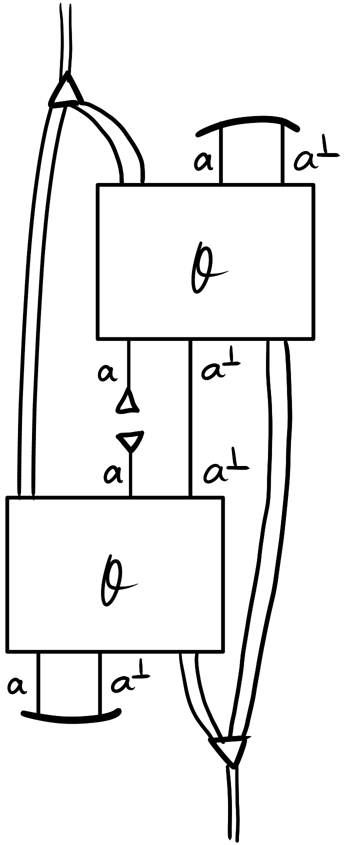




$$
\begin{array}{c}
B \wedge {\scriptstyle \mathsf{ai}\downarrow}\dfrac{\top}{a \vee a^\perp} \\
\Theta \Big\| \\
{\scriptstyle s}\dfrac{B \wedge \left({\scriptstyle =}\dfrac{{\scriptstyle \mathsf{aw}\uparrow}\dfrac{a}{\top} \wedge a^\perp}{{\scriptstyle \mathsf{aw}\downarrow}\dfrac{\bot}{a} \vee a^\perp} \vee C \right)}{(B \wedge (a \vee a^\perp)) \vee C} \\
\Theta \Big\| \\
{\scriptstyle \mathsf{ai}\uparrow}\dfrac{a \wedge a^\perp}{\bot} \vee C
\end{array}
$$

Breaking paths in atomic flows





$$=,\mathsf{c}\uparrow\,\dfrac{B}{B \wedge \mathsf{ai}\!\downarrow\dfrac{\top}{a \vee a^{\perp}}}$$

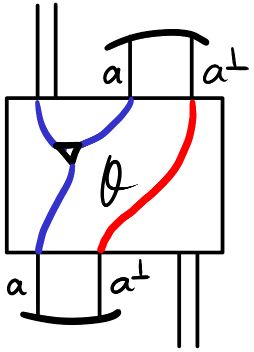
$$\Theta \Big\Vert$$

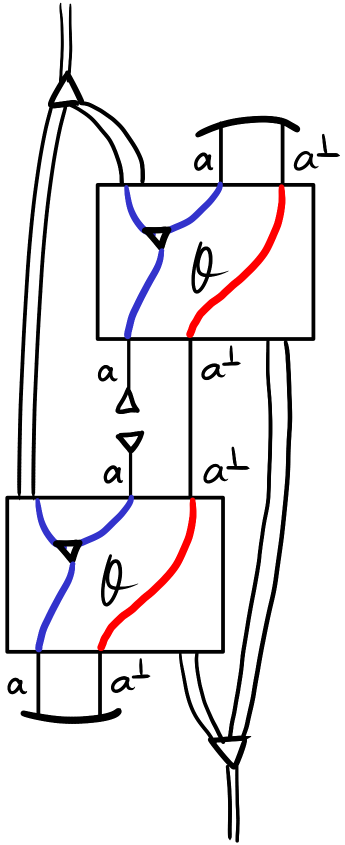
$$\mathsf{s}\,\dfrac{B \wedge \left(=\dfrac{\mathsf{aw}\!\uparrow\dfrac{a}{\top} \wedge a^{\perp}}{\mathsf{aw}\!\downarrow\dfrac{\bot}{a} \vee a^{\perp}} \vee C \right)}{(B \wedge (a \vee a^{\perp})) \vee C}$$

$$\Theta \Big\Vert$$

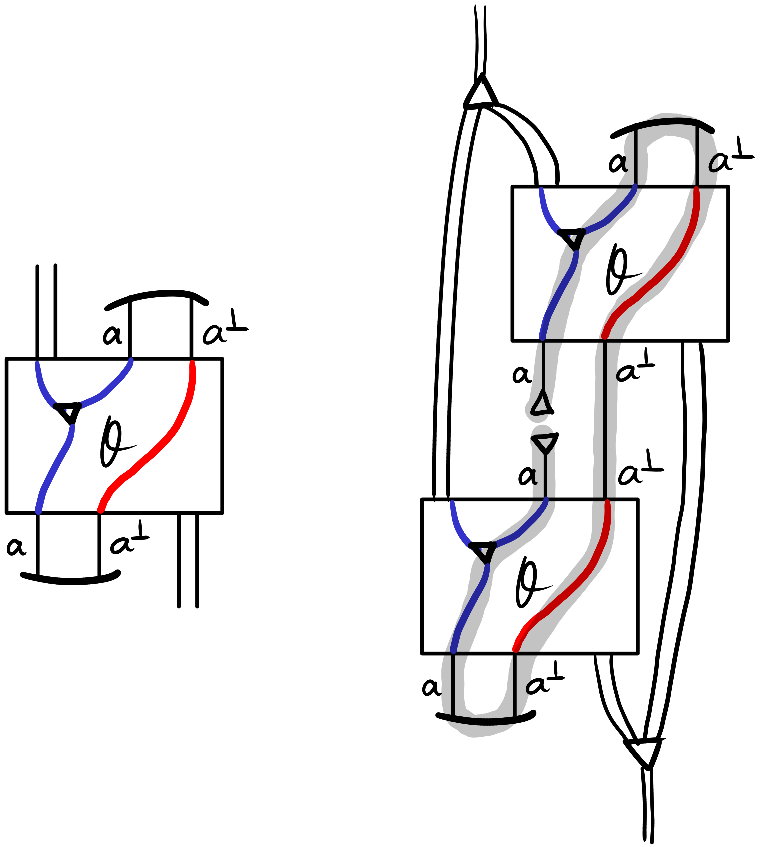
$$=,\mathsf{c}\downarrow\,\dfrac{\mathsf{ai}\!\uparrow\dfrac{a \wedge a^{\perp}}{\bot} \vee C}{C}$$

Breaking paths in atomic flows

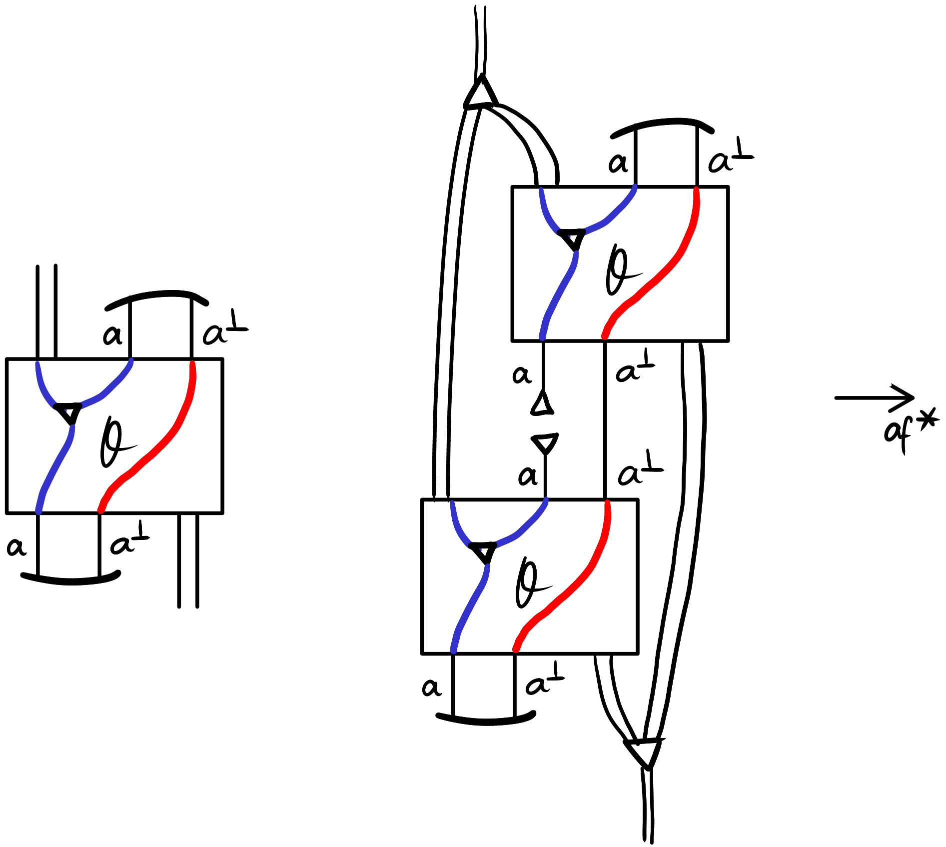


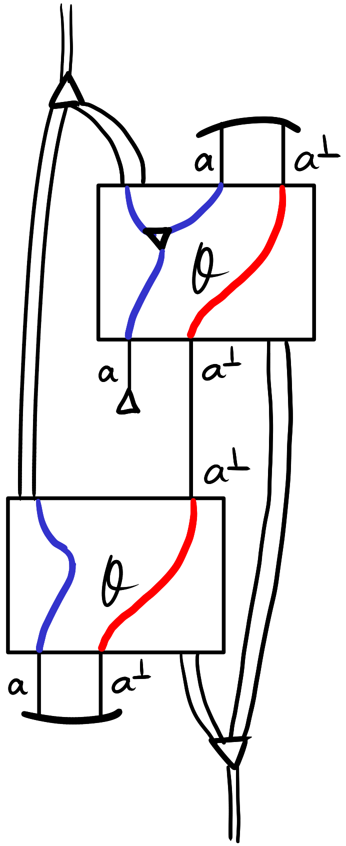


Breaking paths in atomic flows

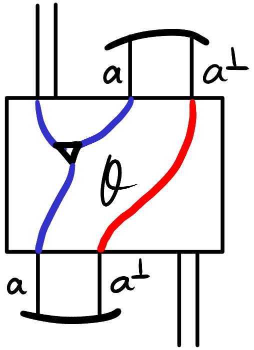


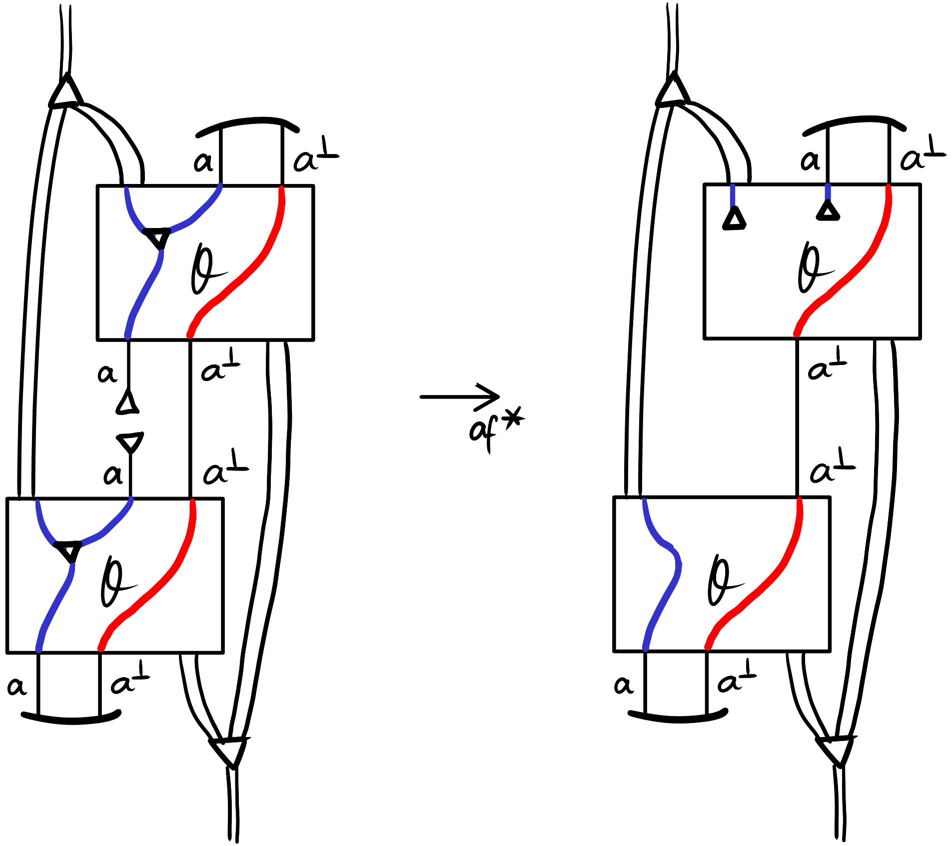
Breaking paths in atomic flows





Breaking paths in atomic flows





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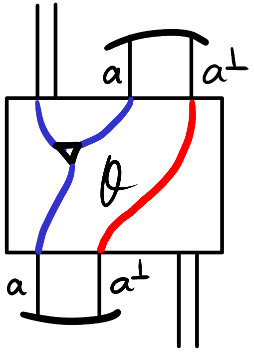
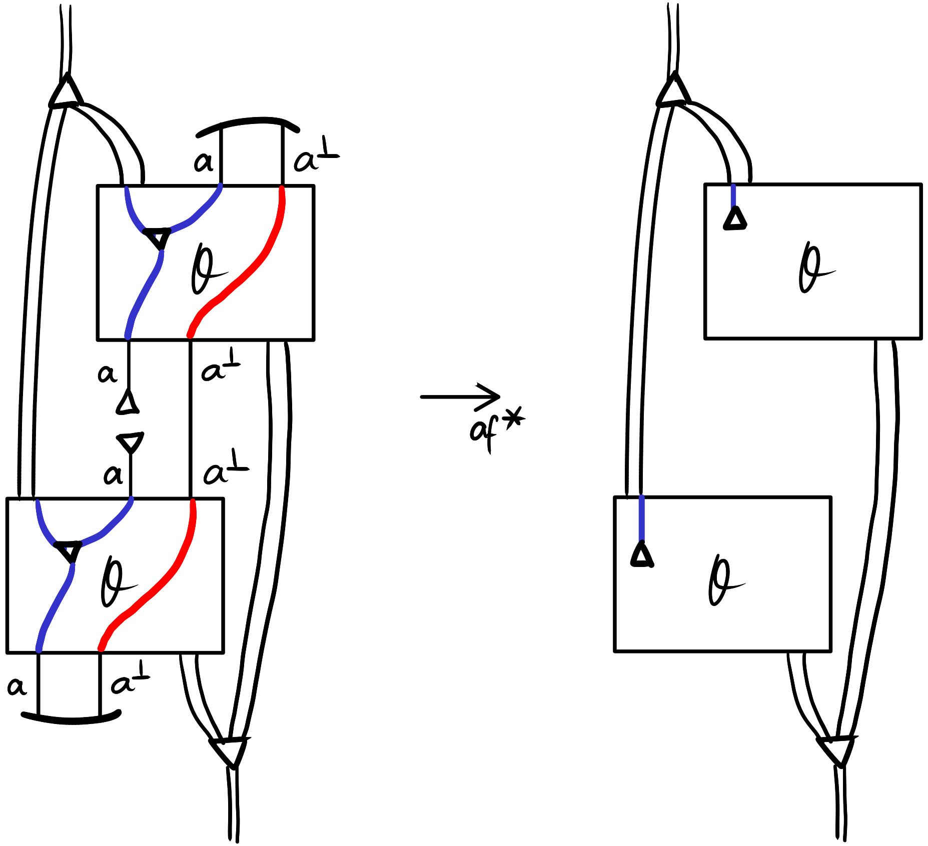
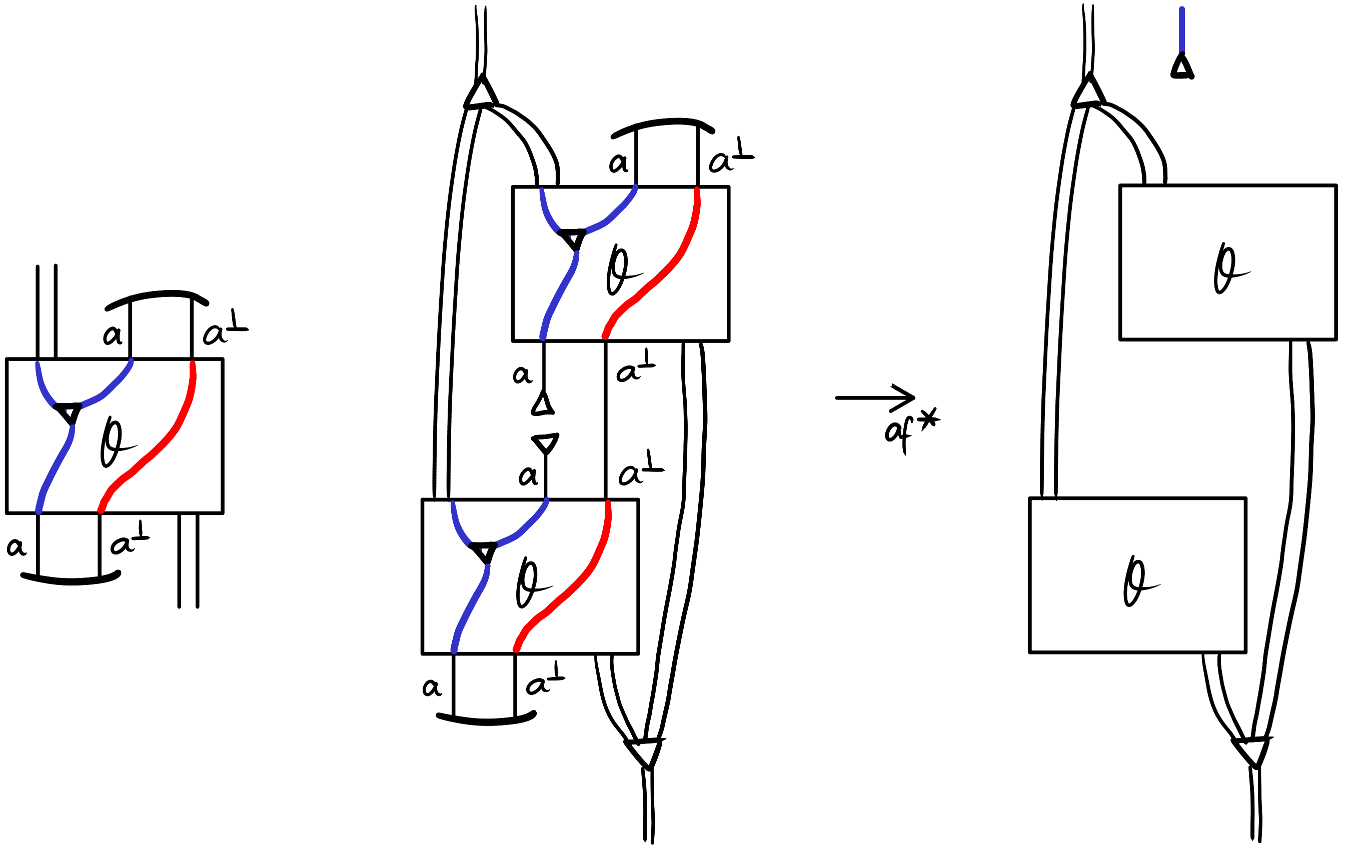
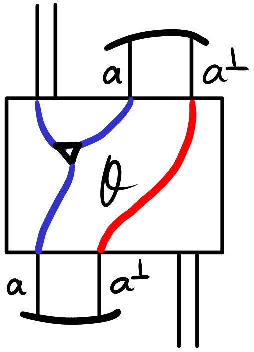


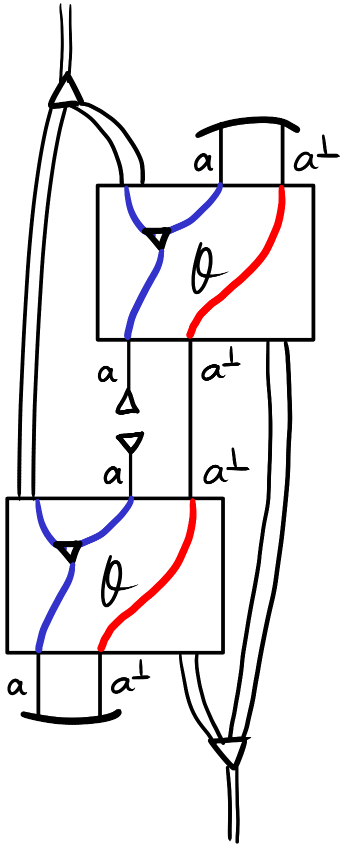


Breaking paths in atomic flows



Breaking paths in atomic flows





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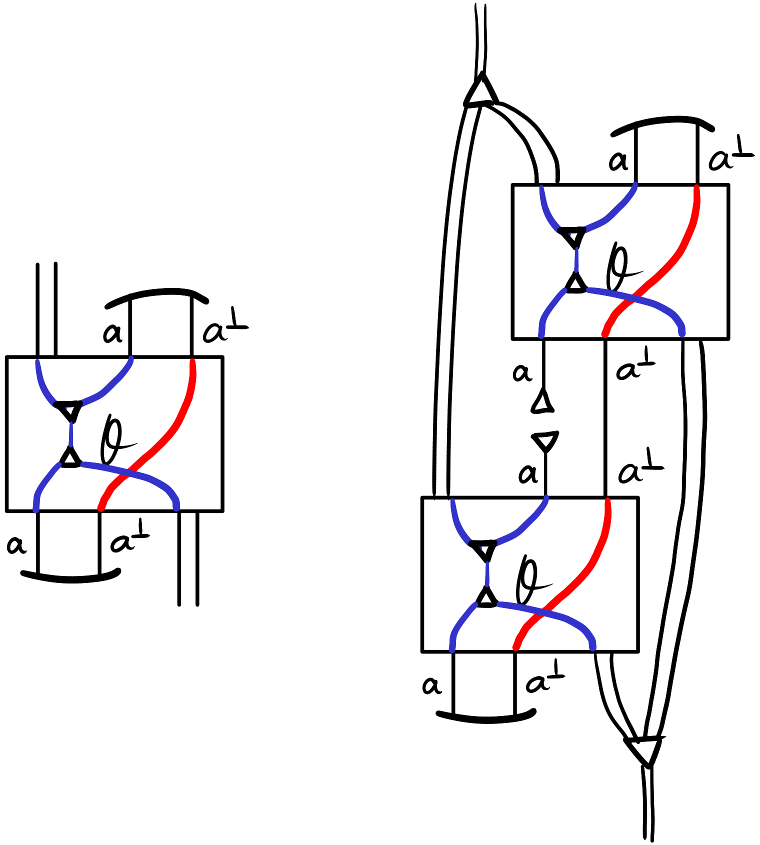
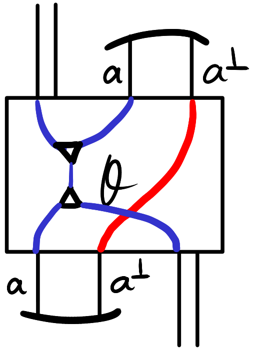
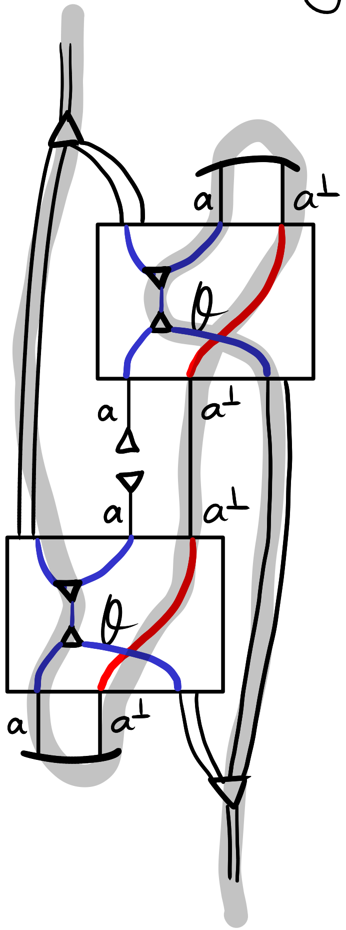
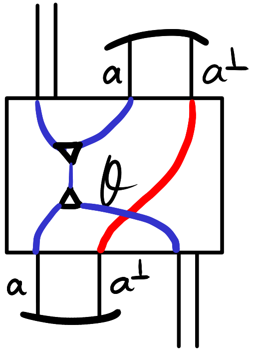


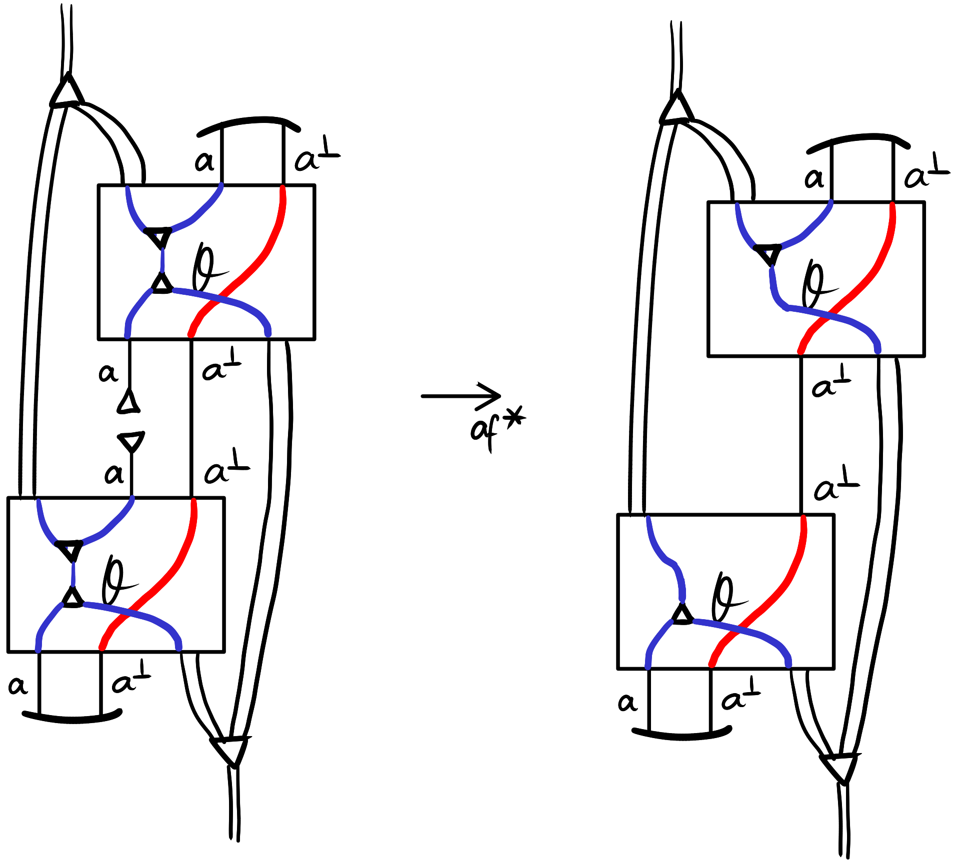
Breaking paths in atomic flows





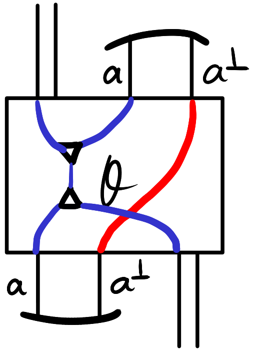
Breaking paths in atomic flows

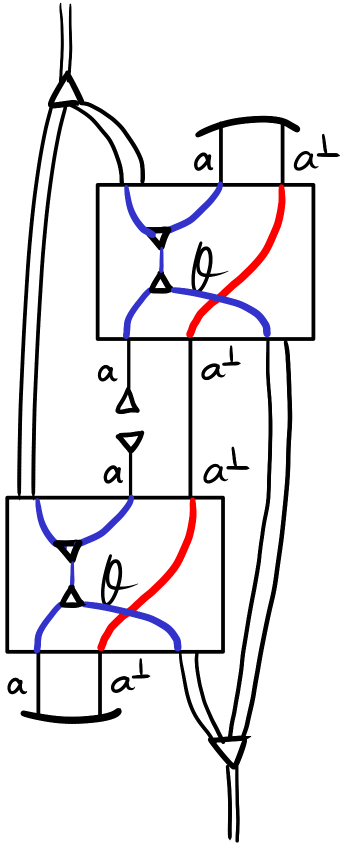




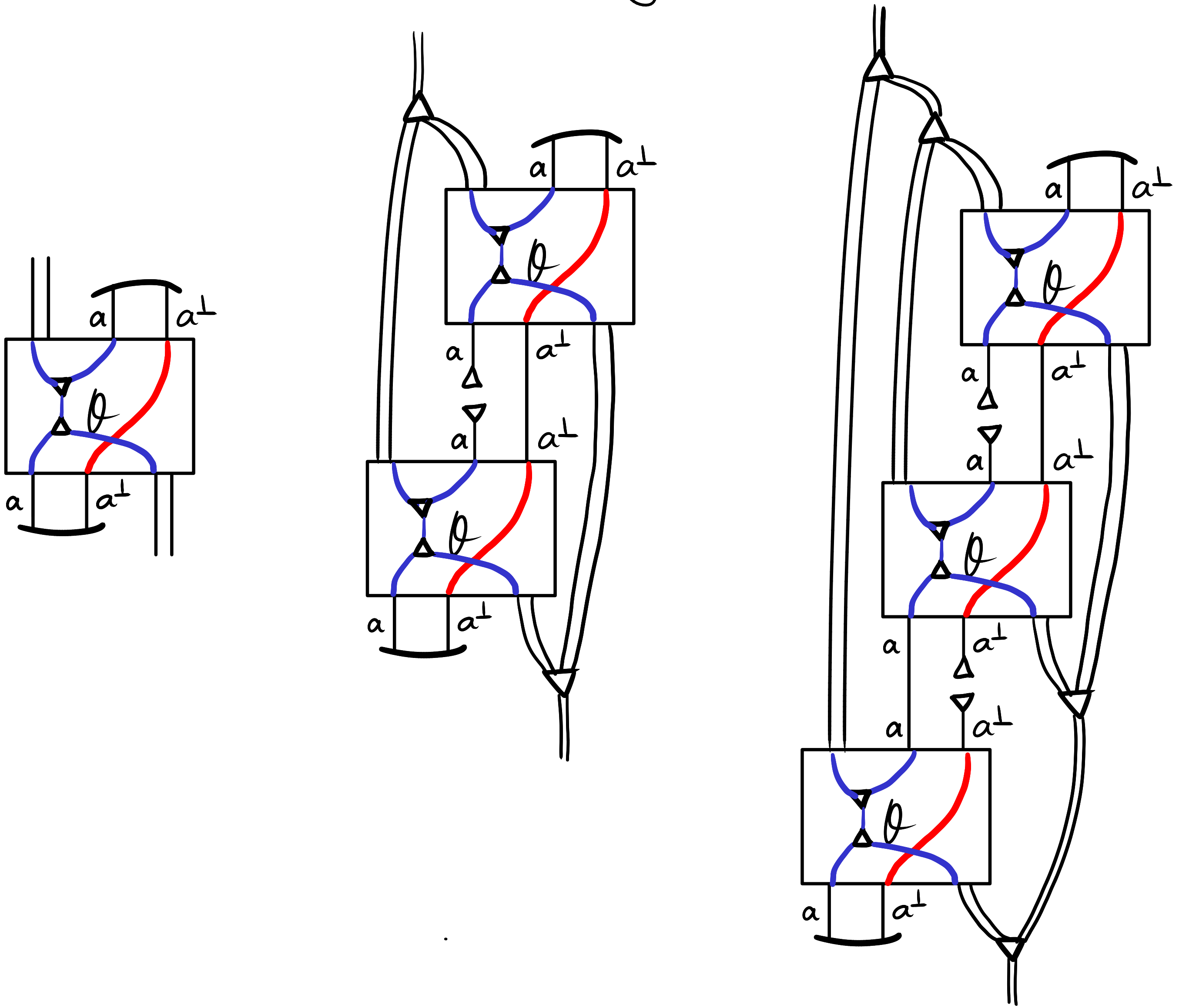
No redexes

Breaking paths in atomic flows

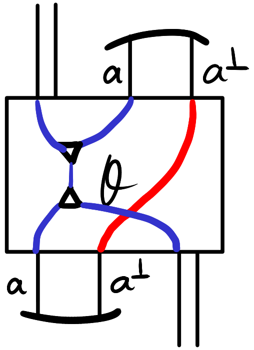


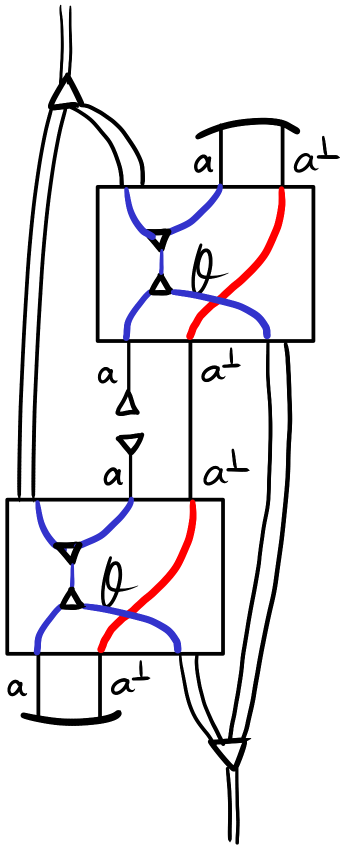


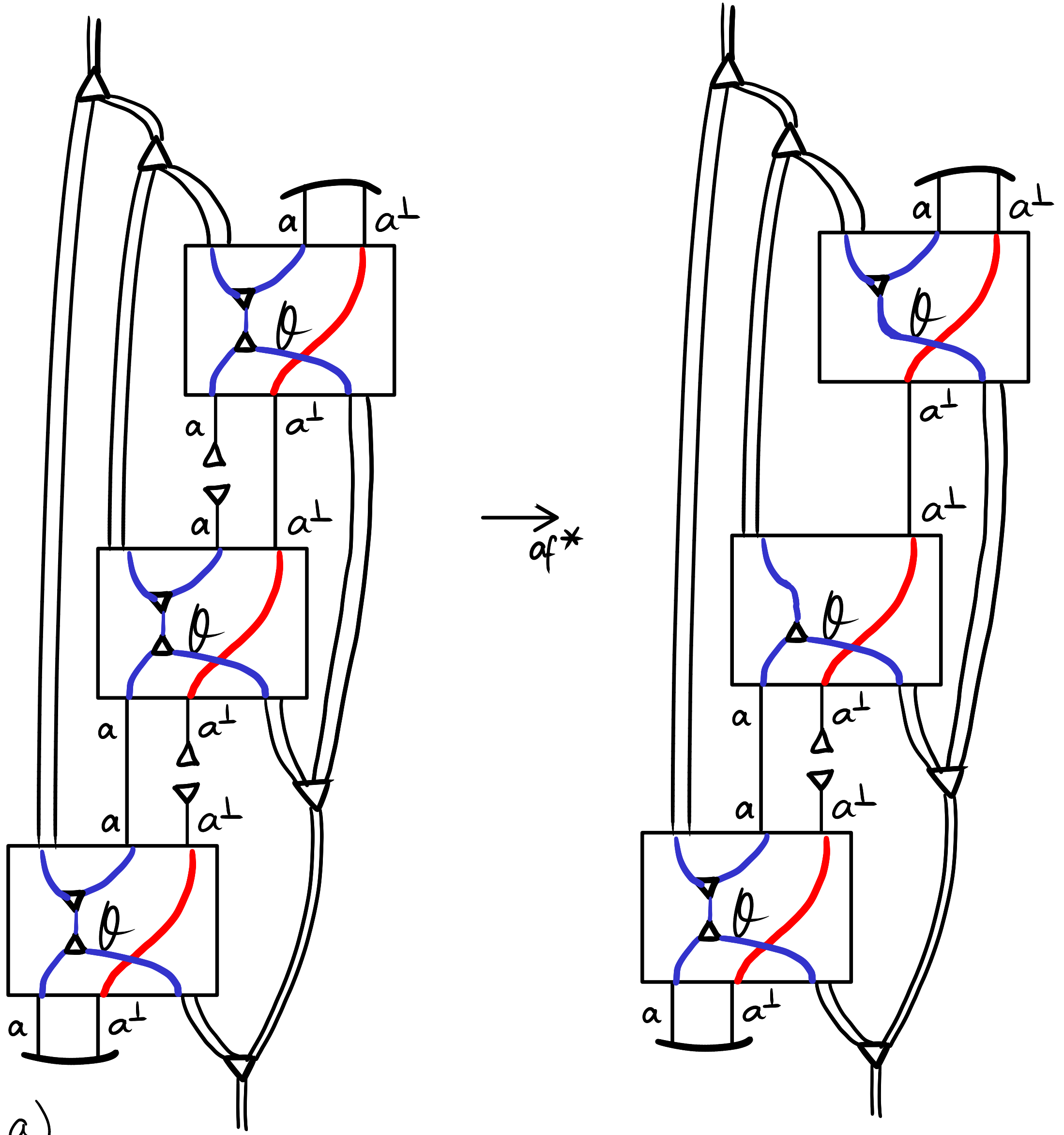
Breaking paths in atomic flows



Breaking paths in atomic flows



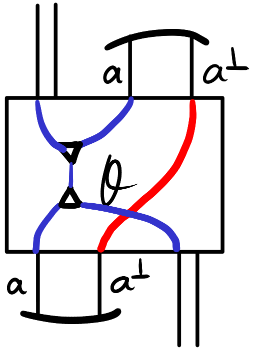


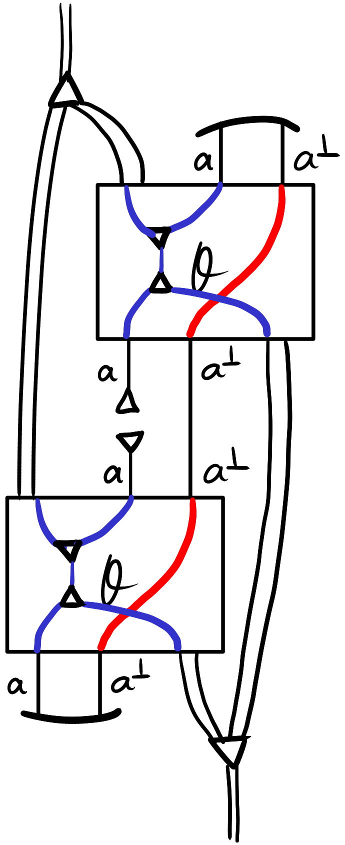


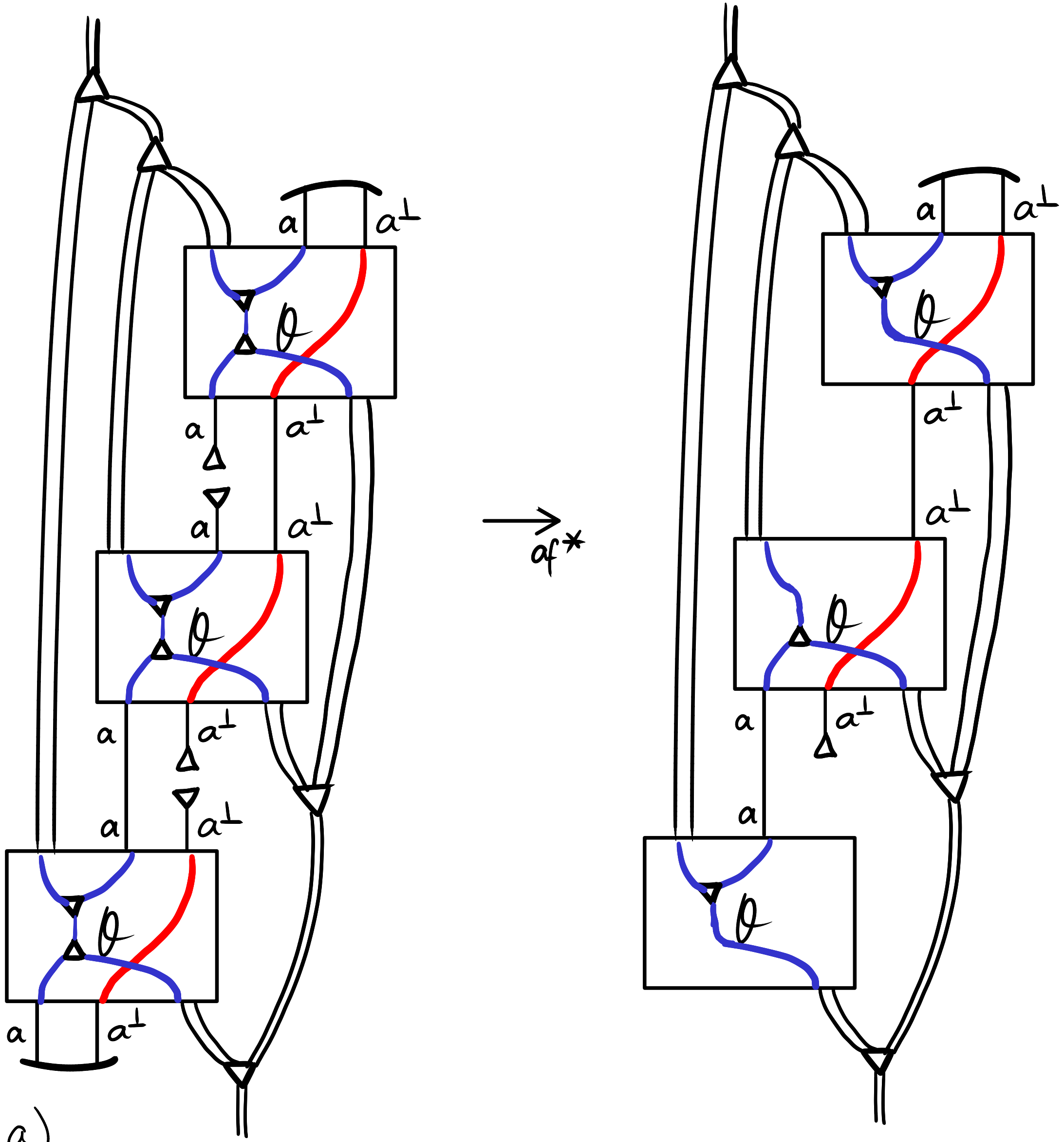
There are no more paths between cuts and identities so flow rewriting will terminate

(for a)

Breaking paths in atomic flows







There are no more paths between cuts and identities so flow rewriting will terminate

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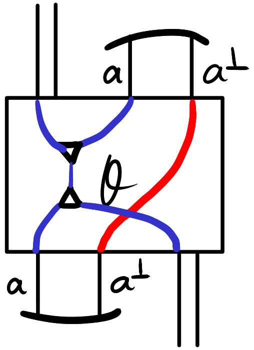
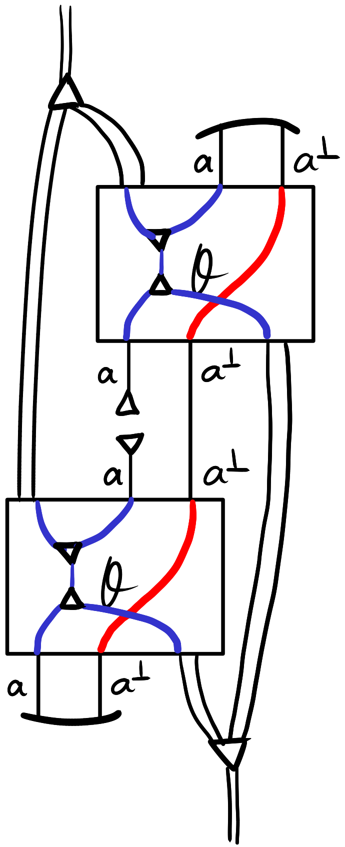
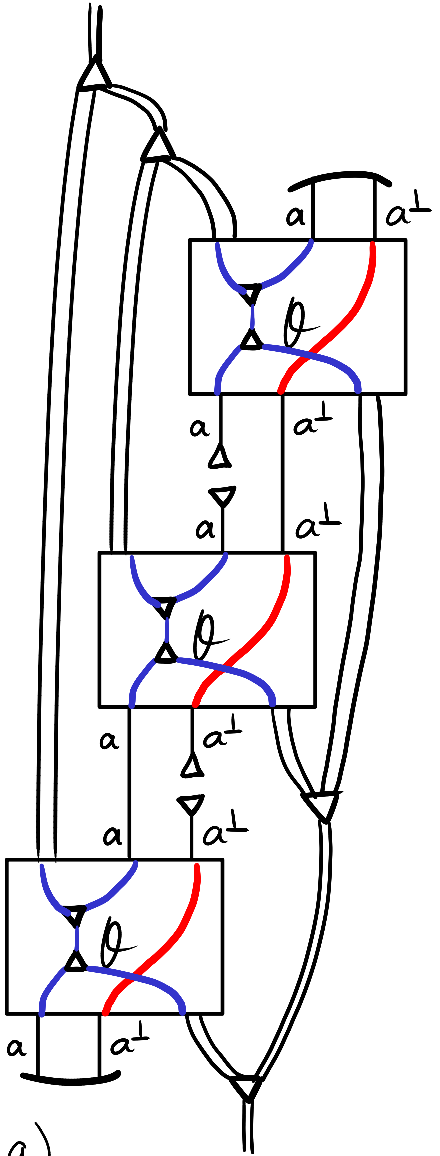
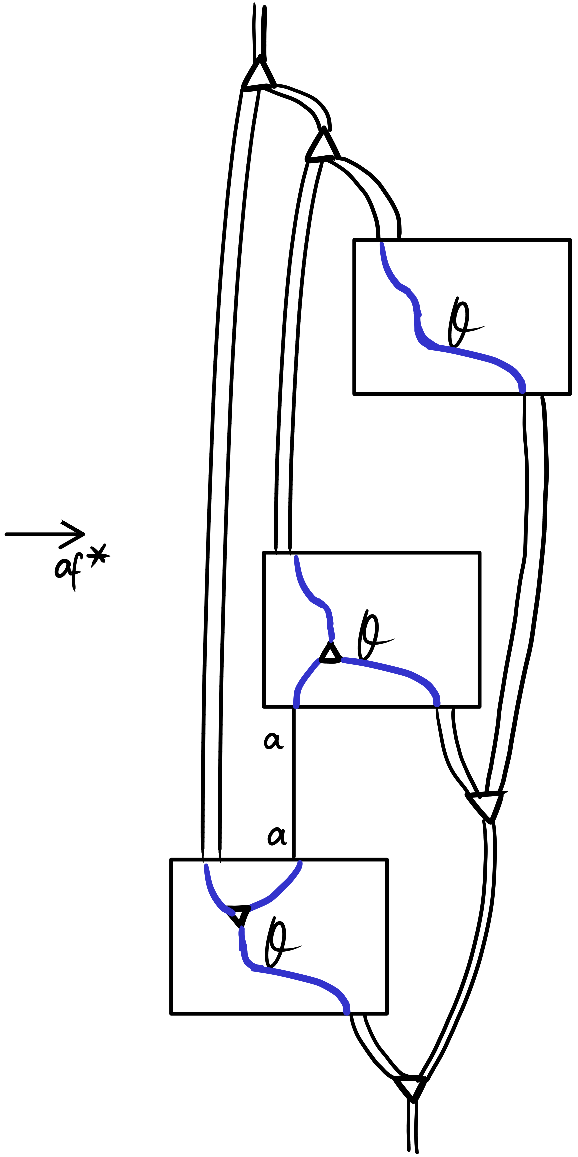
Breaking paths in atomic flows



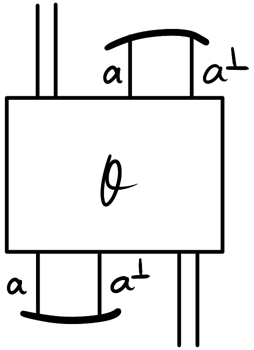


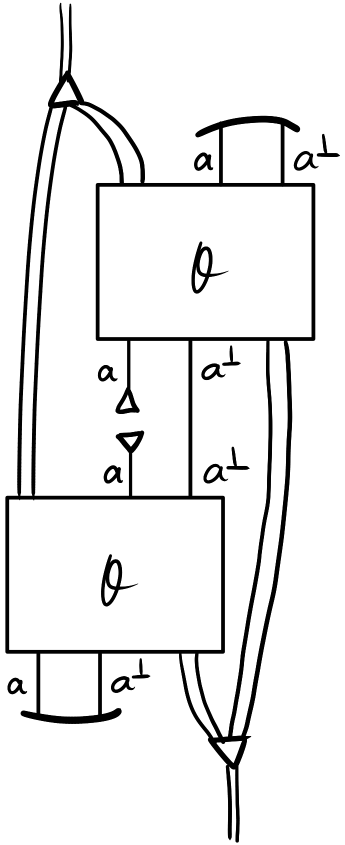


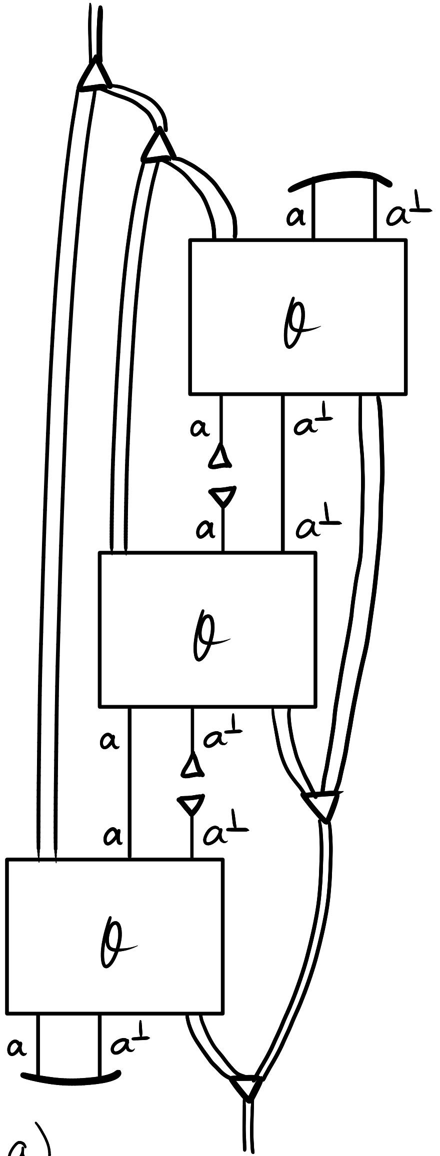


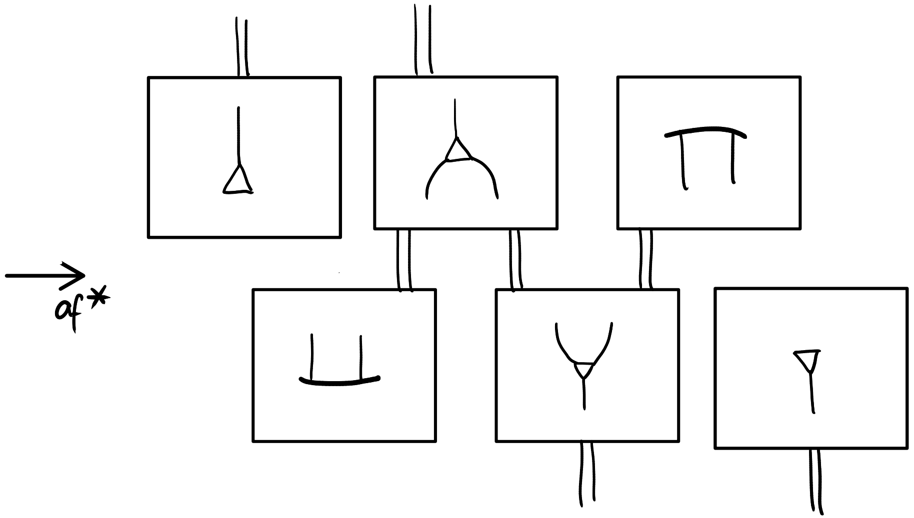
There are no more paths between cuts and identities so flow rewriting will terminate (for a)

Breaking paths in atomic flows



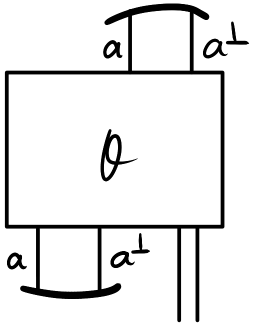


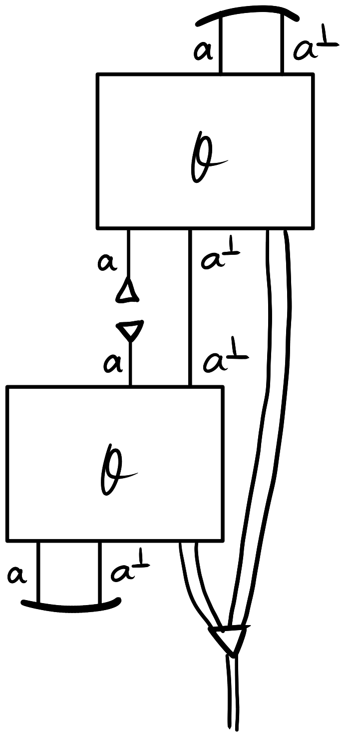


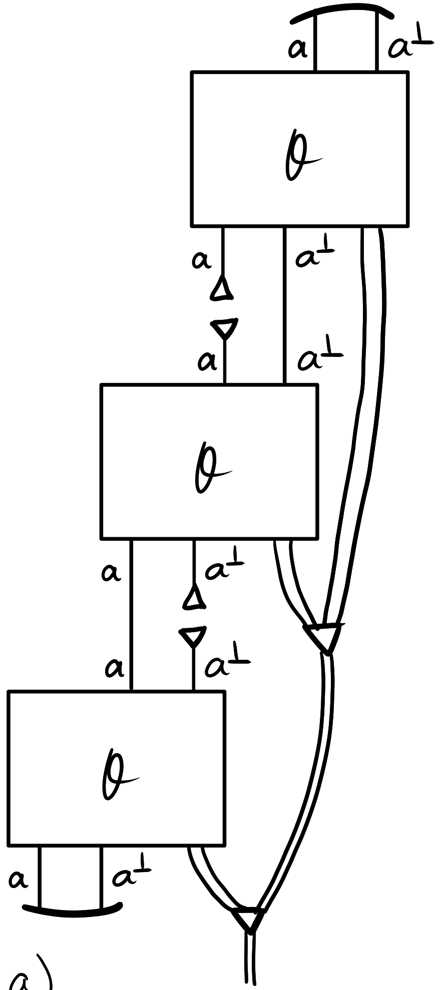


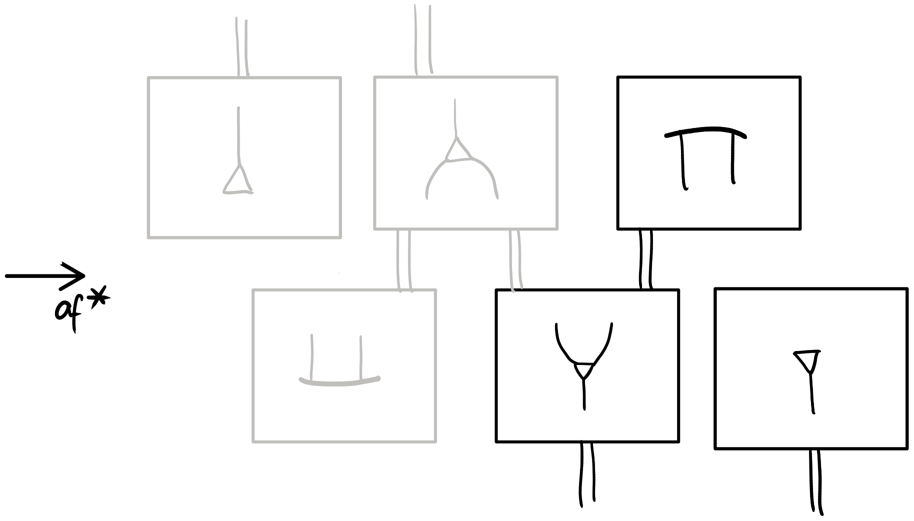
There are no more paths between cuts and identities so flow rewriting will terminate (for a)

Breaking paths in atomic flows









There are no more paths
between cuts and identities
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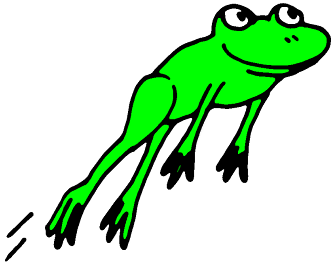
(for a)



Splitting and decomposition

$$i{\uparrow}\,\frac{A \otimes B \otimes (A^{\perp} \parr B^{\perp})}{\perp} \quad \xrightarrow[i{\uparrow} \,\to\, ai{\uparrow}]{} \quad {}_{=}\frac{A \otimes {}_{s}\dfrac{B \otimes (A^{\perp} \parr B^{\perp})}{A^{\perp} \parr {}_{i\uparrow}\dfrac{B \otimes B^{\perp}}{\perp}}}{{}_{i\uparrow}\dfrac{A \otimes A^{\perp}}{\perp}}$$

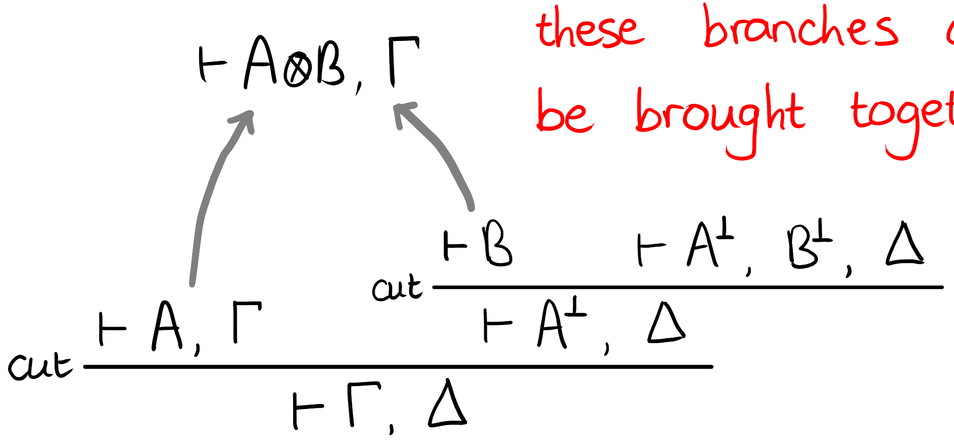
Cut cannot be made atomic in the sequent calculus

$$\mathrm{cut}\,\frac{\vdash A \otimes B\, ,\ \Gamma \qquad \vdash A^{\perp} \parr B^{\perp},\ \Delta}{\vdash \Gamma, \Delta}$$

$$i{\uparrow}\,\frac{A \otimes B \otimes (A^{\perp} \parr B^{\perp})}{\bot} \quad \xrightarrow[\quad i{\uparrow}\ \to\ ai{\uparrow}\quad]{} \quad = \frac{A \otimes \ \mathrm{s}\,\dfrac{B \otimes (A^{\perp} \parr B^{\perp})}{A^{\perp} \parr \ i{\uparrow}\,\dfrac{B \otimes B^{\perp}}{\bot}}}{i{\uparrow}\,\dfrac{A \otimes A^{\perp}}{\bot}}$$

Cut cannot be made atomic in the sequent calculus

$$\text{cut}\,\frac{\vdash A\otimes B\,,\,\Gamma \qquad\qquad \vdash A^\perp \parr B^\perp, \,\Delta}{\vdash \Gamma, \Delta}$$



$$\vdash A\otimes B, \Gamma$$

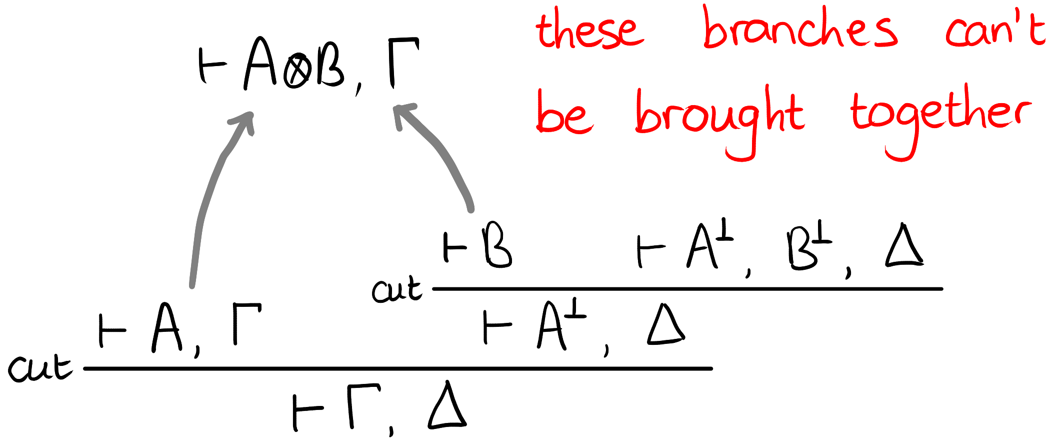
$$\text{cut}\,\frac{\vdash A, \Gamma \qquad \text{cut}\,\dfrac{\vdash B \qquad \vdash A^\perp, B^\perp, \Delta}{\vdash A^\perp, \Delta}}{\vdash \Gamma, \Delta}$$

these branches can't be brought together

$$\text{i}\!\uparrow\frac{A\otimes B \otimes (A^\perp \parr B^\perp)}{\perp} \qquad \xrightarrow[\text{i}\uparrow \,\to\, \text{ai}\uparrow]{} \qquad =\,\frac{A \otimes \text{s}\,\dfrac{B\otimes(A^\perp \parr B^\perp)}{A^\perp \parr \text{i}\!\uparrow\dfrac{B\otimes B^\perp}{\perp}}}{\text{i}\!\uparrow\dfrac{A\otimes A^\perp}{\perp}}$$

Cut cannot be made atomic in the sequent calculus

$$\text{cut}\,\frac{\vdash A\otimes B,\ \Gamma \qquad \vdash A^{\perp}\parr B^{\perp},\ \Delta}{\vdash \Gamma,\ \Delta}$$



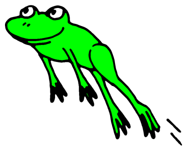
$$\vdash A\otimes B,\ \Gamma$$

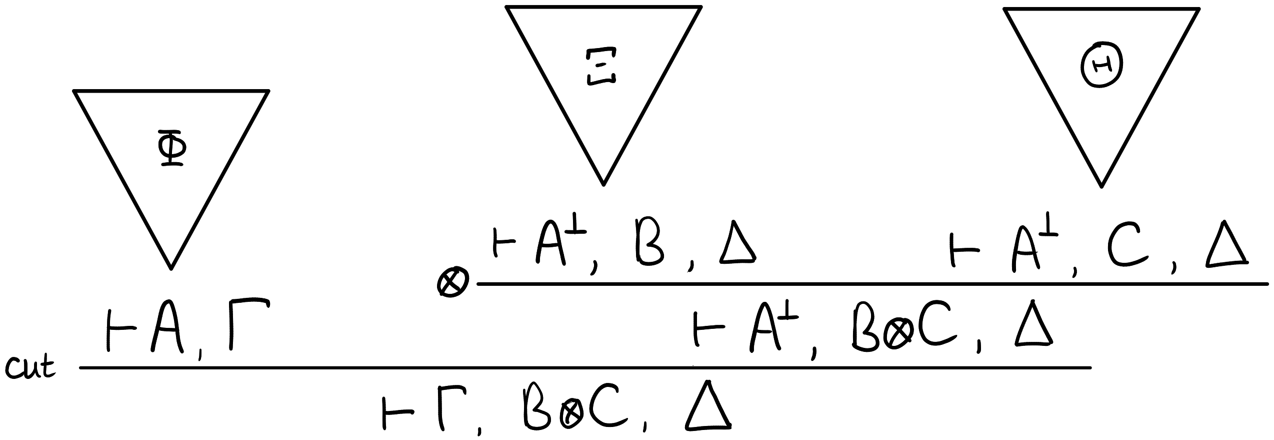
$$\text{cut}\,\frac{\vdash A,\ \Gamma \qquad \text{cut}\,\dfrac{\vdash B \qquad \vdash A^{\perp},\ B^{\perp},\ \Delta}{\vdash A^{\perp},\ \Delta}}{\vdash \Gamma,\ \Delta}$$

these branches can't be brought together

Splitting of the context is guaranteed by the formalism in the sequent calculus, but not in deep inference

$$\mathsf{i}{\uparrow}\,\frac{A\otimes B \otimes (A^{\perp}\parr B^{\perp})}{\perp} \qquad \xrightarrow[\mathsf{i}{\uparrow}\ \to\ \mathsf{ai}{\uparrow}]{} \qquad =\,\frac{A \otimes\ \mathsf{s}\,\dfrac{B\otimes(A^{\perp}\parr B^{\perp})}{A^{\perp}\parr\ \mathsf{i}{\uparrow}\,\dfrac{B\otimes B^{\perp}}{\perp}}}{\mathsf{i}{\uparrow}\,\dfrac{A\otimes A^{\perp}}{\perp}}$$





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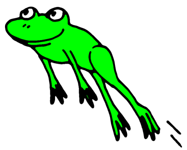
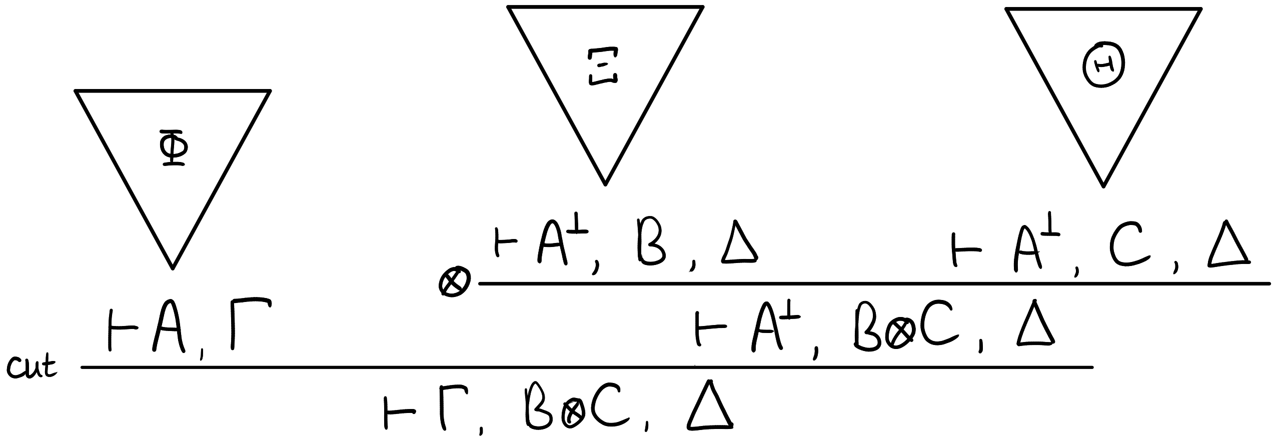
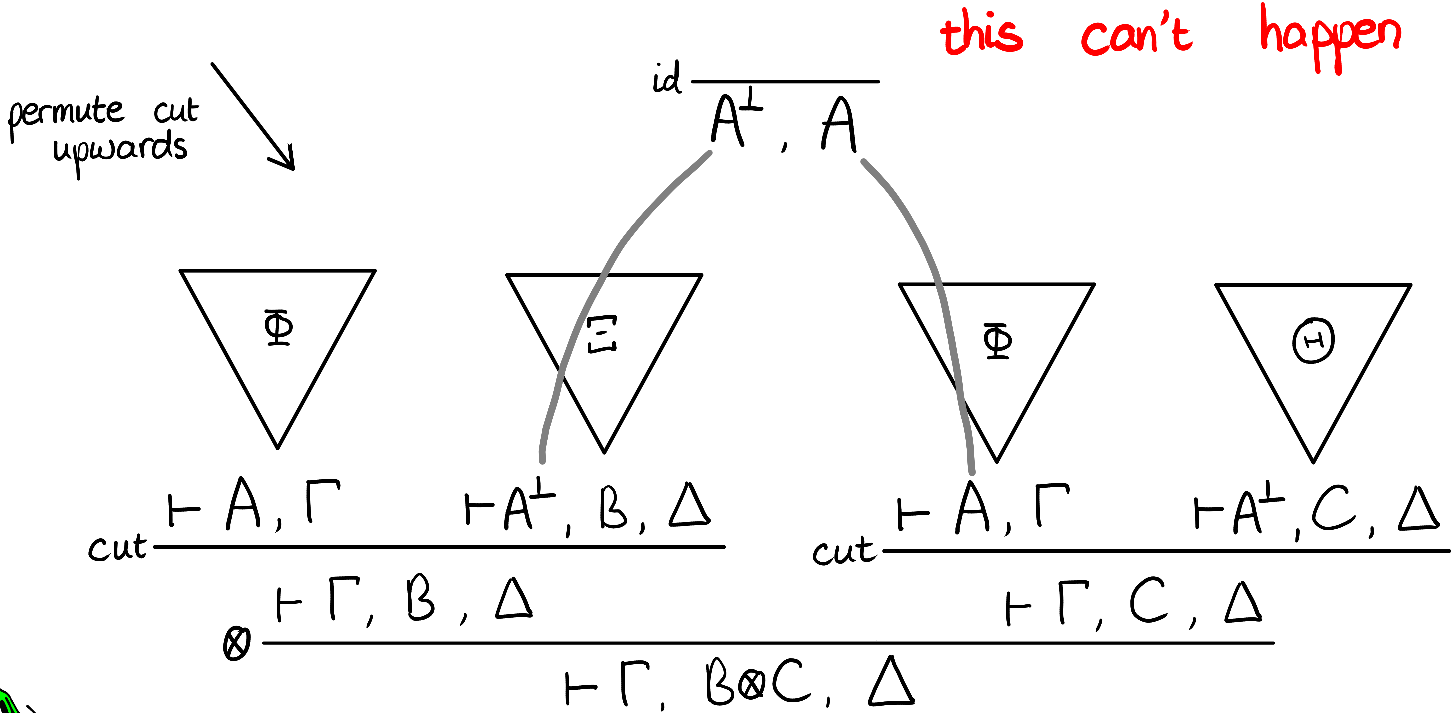
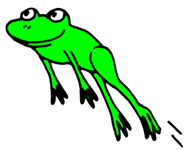


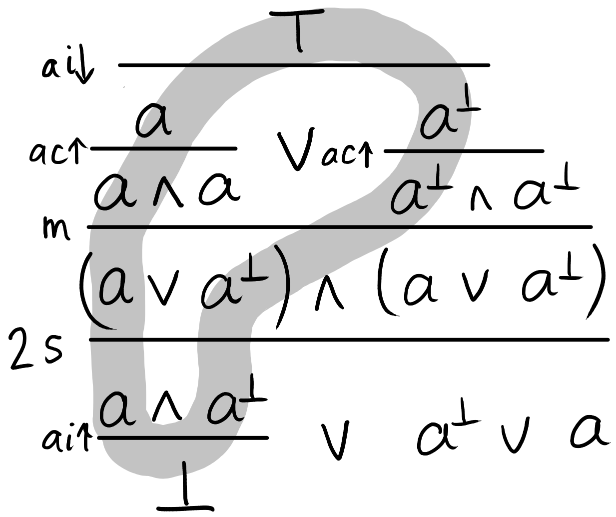




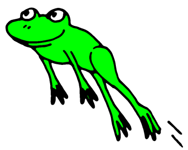
this can't happen

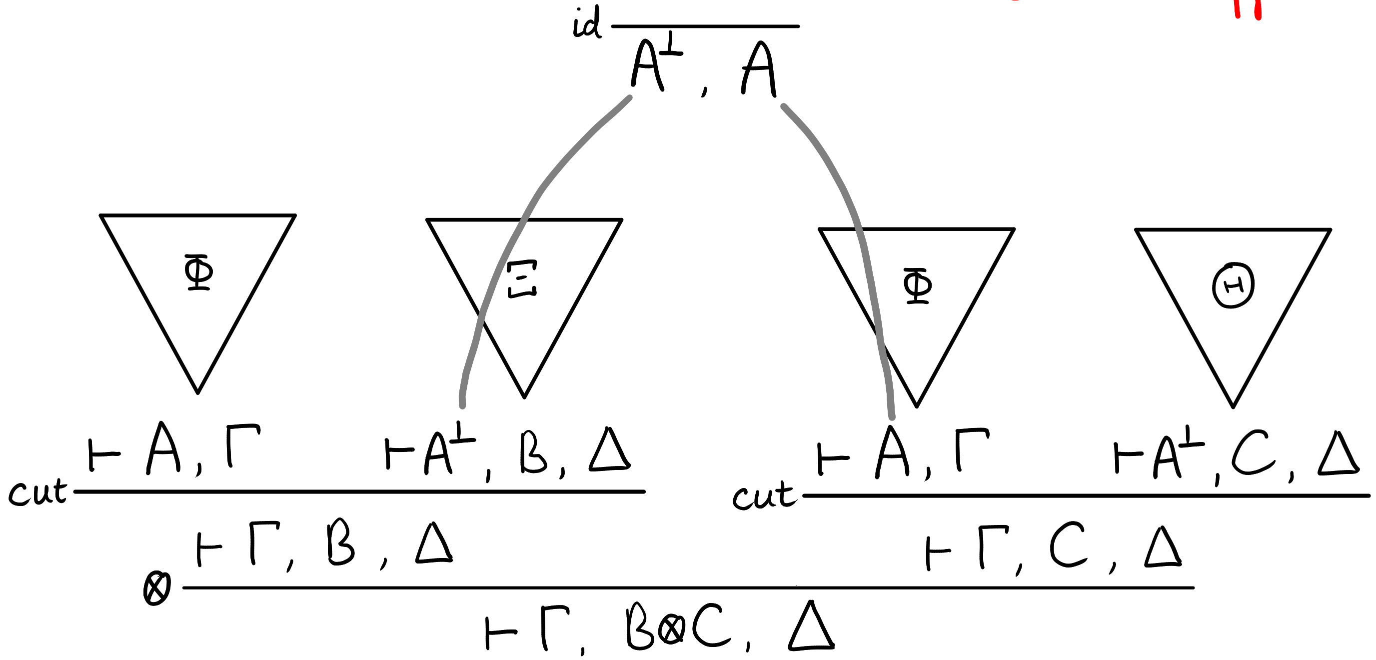
Splitting of the context is guaranteed by the formalism in the sequent calculus, but not in deep inference





Splitting of the context is guaranteed by the formalism in the sequent calculus, but not in deep inference





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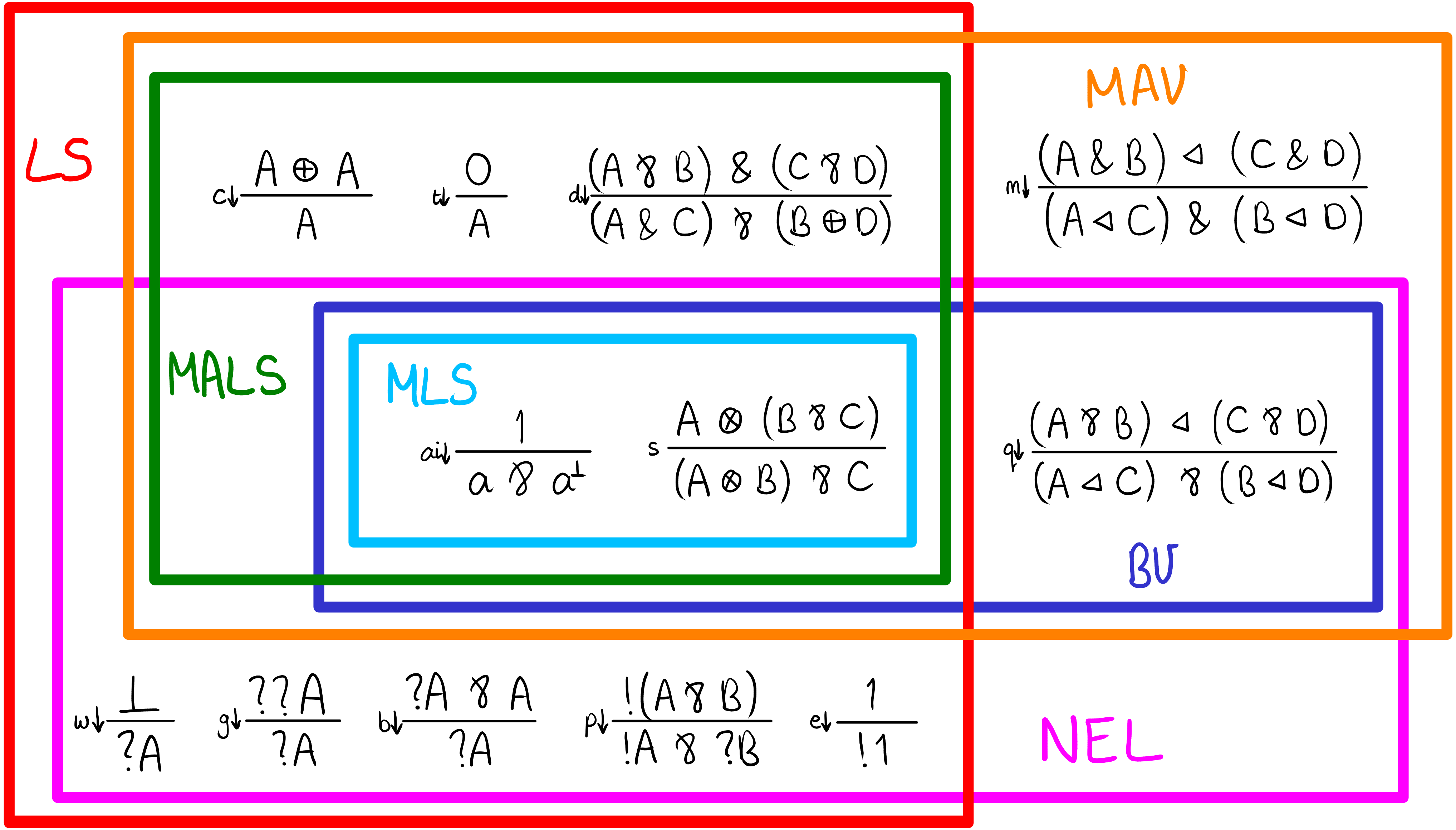

LS
MAV
MALS
MLS
BV
NEL
$c\downarrow\dfrac{A \oplus A}{A}$
$t\downarrow\dfrac{0}{A}$
$d\downarrow\dfrac{(A \parr B) \,\&\, (C \parr D)}{(A \,\&\, C) \parr (B \oplus D)}$
$m\downarrow\dfrac{(A \,\&\, B) \lhd (C \,\&\, D)}{(A \lhd C) \,\&\, (B \lhd D)}$
$ai\downarrow\dfrac{1}{a \parr a^{\perp}}$
$s\,\dfrac{A \otimes (B \parr C)}{(A \otimes B) \parr C}$
$q\downarrow\dfrac{(A \parr B) \lhd (C \parr D)}{(A \lhd C) \parr (B \lhd D)}$
$w\downarrow\dfrac{\perp}{?A}$
$g\downarrow\dfrac{??A}{?A}$
$b\downarrow\dfrac{?A \parr A}{?A}$
$p\downarrow\dfrac{!(A \parr B)}{!A \parr ?B}$
$e\downarrow\dfrac{1}{!1}$

$$\boxed{\mathsf{ai}{\downarrow}\,\frac{1}{a \parr a^{\perp}} \qquad \mathsf{s}\,\frac{A \otimes (B \parr C)}{(A \otimes B) \parr C}}$$

MLS

This example is to show the *simplest possible* case of a cut in an MLS+cut proof, it is not intended as a general case, but a best case scenario to show what is intended by the splitting procedure

$$\mathsf{ai}{\uparrow}\,\frac{a \otimes a^{\perp}}{\perp} \qquad \overset{\Pi}{\parr} \qquad B$$

$$\boxed{\mathsf{ai}\!\downarrow\frac{1}{a \parr a^{\perp}} \qquad \mathsf{s}\,\frac{A \otimes (B \parr C)}{(A \otimes B) \parr C}}$$

MLS

$$\mathbb{T}$$

$$\mathsf{ai}\!\uparrow\frac{a \otimes a^{\perp}}{\bot} \qquad \parr \qquad \begin{array}{c} K\{a^{\perp}\} \parr H\{a\} \\ \Delta \Vert \\ B \end{array}$$

$$\boxed{\text{ai}\!\downarrow\frac{1}{a \parr a^{\perp}} \qquad \mathsf{s}\,\frac{A \otimes (B \parr C)}{(A \otimes B) \parr C}}$$

MLS

$$=,2\mathsf{s}\,\frac{\begin{array}{c}\mathbb{T}\\ a \parr K\{a^{\perp}\}\end{array} \quad\otimes\quad \begin{array}{c}\mathbb{T}\\ a^{\perp} \parr H\{a\}\end{array}}{\text{ai}\!\uparrow\frac{a \otimes a^{\perp}}{\perp} \quad\parr\quad \begin{array}{c}K\{a^{\perp}\} \parr H\{a\}\\ \Delta \,\Big\| \\ B\end{array}}$$

$$\boxed{\mathsf{ai}{\downarrow}\,\frac{1}{a \parr a^{\perp}} \qquad \mathsf{s}\,\frac{A \otimes (B \parr C)}{(A \otimes B) \parr C}}$$

MLS

$$=,2\mathsf{s}\;\frac{\begin{array}{c} \mathbb{T} \\ \Vert \\ a \parr \quad P \otimes a^{\perp} \\ \Omega\Vert \\ K\{a^{\perp}\} \end{array} \quad \otimes \quad \begin{array}{c} \mathbb{T} \\ \Vert \\ a^{\perp} \parr \; H\{a\} \end{array}}{\mathsf{ai}{\uparrow}\,\frac{a \otimes a^{\perp}}{\perp} \quad \parr \quad \begin{array}{c} K\{a^{\perp}\} \parr H\{a\} \\ \Delta\Vert \\ B \end{array}}$$

$$\boxed{{ai\downarrow}\,\dfrac{1}{a \parr a^{\perp}} \qquad {s}\,\dfrac{A \otimes (B \parr C)}{(A \otimes B) \parr C}}$$

MLS

$$
{=,2s}\,\dfrac{a \parr \begin{array}{c} \pi\,\overline{\Vert} \\ {=,s}\,\dfrac{{ai\downarrow}\,\dfrac{1}{a \parr a^{\perp}} \otimes \rho}{\rho \otimes a^{\perp}} \\ \Omega \Vert \\ K\{a^{\perp}\} \end{array} \otimes \begin{array}{c} \overline{\Vert} \\ a^{\perp} \parr H\{a\} \end{array}}{{ai\uparrow}\,\dfrac{a \otimes a^{\perp}}{\bot} \parr \begin{array}{c} K\{a^{\perp}\} \parr H\{a\} \\ \Delta \Vert \\ B \end{array}}
$$

$$\boxed{\mathsf{ai}\!\downarrow\frac{1}{a \parr a^\perp} \qquad \mathsf{s}\,\frac{A \otimes (B \parr C)}{(A \otimes B) \parr C}}$$

MLS

In this best case scenario, we are able to extract the pieces of the proof that are necessary to rebuild a cut-free proof of the same conclusion

$$=,2\mathsf{s}\,\frac{\begin{array}{c} \Pi \Big\| \\ =,\mathsf{s}\,\dfrac{\mathsf{ai}\!\downarrow\dfrac{1}{a \parr a^\perp} \otimes P}{P \otimes a^\perp} \\ a \parr \quad \Omega \Big\| \\ K\{a^\perp\} \end{array} \quad \otimes \quad \begin{array}{c} \Psi \Big\| \\ \mathsf{s}\,\dfrac{\mathsf{ai}\!\downarrow\dfrac{1}{a^\perp \parr a} \otimes Q}{a \otimes Q} \\ a^\perp \parr \quad \Theta \Big\| \\ H\{a\} \end{array}}{\mathsf{ai}\!\uparrow\dfrac{a \otimes a^\perp}{\perp} \quad \parr \quad \begin{array}{c} K\{a^\perp\} \parr H\{a\} \\ \Delta \Big\| \\ B \end{array}} \quad \longrightarrow \quad \begin{array}{c} 2\mathsf{s}\,\dfrac{\begin{array}{c}\Pi \Big\| \\ P\end{array} \otimes \mathsf{ai}\!\downarrow\dfrac{1}{a^\perp \parr a} \otimes \begin{array}{c}\Psi \Big\| \\ Q\end{array}}{\begin{array}{c} P \otimes a^\perp \\ \Omega \Big\| \\ K\{a^\perp\} \end{array} \parr \begin{array}{c} a \otimes Q \\ \Theta \Big\| \\ H\{a\} \end{array}} \\ \Delta \Big\| \\ B \end{array}$$

$$\boxed{\text{ai}\downarrow\frac{1}{a \parr a^{\perp}} \qquad \text{s}\,\frac{A \otimes (B \parr C)}{(A \otimes B) \parr C}}$$

MLS

If $\begin{array}{c}\Vert\,\text{MLS}\\ (A \otimes B) \parr C\end{array}$ then there exist K_A, K_B such that $\begin{array}{c}K_A \parr K_B\\ \Vert\,\text{MLS}\\ K\end{array}$, $\begin{array}{c}\Vert\,\text{MLS}\\ A \parr K_A\end{array}$, $\begin{array}{c}\Vert\,\text{MLS}\\ B \parr K_B\end{array}$

If $\begin{array}{c}\Vert\,\text{MLS}\\ a \parr C\end{array}$ then $\begin{array}{c}a^{\perp}\\ \Vert\,\text{MLS}\\ C\end{array}$

This is the theorem that we need to prove in order to extend the technique to more general cases

$$\text{=,2s}\,\frac{\left(\text{=,s}\,\dfrac{\text{ai}\downarrow\dfrac{1}{a \parr a^{\perp}} \otimes \begin{array}{c}\Pi\,\Vert\\ P\end{array}}{a \parr \begin{array}{c}P \otimes a^{\perp}\\ \Omega\,\Vert\\ K\{a^{\perp}\}\end{array}}\right) \otimes \left(\text{s}\,\dfrac{\text{ai}\downarrow\dfrac{1}{a^{\perp} \parr a} \otimes \begin{array}{c}\Psi\,\Vert\\ Q\end{array}}{a^{\perp} \parr \begin{array}{c}a \otimes Q\\ \Theta\,\Vert\\ H\{a\}\end{array}}\right)}{\text{ai}\uparrow\dfrac{a \otimes a^{\perp}}{\perp} \parr \begin{array}{c}K\{a^{\perp}\} \parr H\{a\}\\ \Delta\,\Vert\\ B\end{array}}
\longrightarrow
\begin{array}{c}\text{2s}\,\dfrac{\begin{array}{c}\Pi\,\Vert\\ P\end{array} \otimes \text{ai}\downarrow\dfrac{1}{a^{\perp} \parr a} \otimes \begin{array}{c}\Psi\,\Vert\\ Q\end{array}}{\begin{array}{c}P \otimes a^{\perp}\\ \Omega\,\Vert\\ K\{a^{\perp}\}\end{array} \parr \begin{array}{c}a \otimes Q\\ \Theta\,\Vert\\ H\{a\}\end{array}}\\ \Delta\,\Vert\\ B\end{array}$$

$$\boxed{\quad \mathsf{ai}\!\downarrow \frac{1}{a \parr a^{\perp}} \qquad \mathsf{s}\,\frac{A \otimes (B \parr C)}{(A \otimes B) \parr C} \quad}$$

MLS

If $\begin{array}{c} \overline{\Vert}\;\mathsf{MLS} \\ (A \otimes B) \parr C \end{array}$ then there exist K_A, K_B such that $\begin{array}{c} K_A \parr K_B \\ \Vert\;\mathsf{MLS} \\ K \end{array}$, $\begin{array}{c} \overline{\Vert}\;\mathsf{MLS} \\ A \parr K_A \end{array}$, $\begin{array}{c} \overline{\Vert}\;\mathsf{MLS} \\ B \parr K_B \end{array}$

If $\begin{array}{c} \overline{\Vert}\;\mathsf{MLS} \\ a \parr C \end{array}$ then $\begin{array}{c} a^{\perp} \\ \Vert\;\mathsf{MLS} \\ C \end{array}$

$$= \frac{\begin{array}{c} \overline{\Vert}\;\mathsf{MLS} \\ =,2\mathsf{s}\,\dfrac{\left(a \parr K\{a^{\perp}\}\right) \otimes \left(a^{\perp} \parr H\{a\}\right)}{\left(a \otimes a^{\perp}\right) \parr K\{a^{\perp}\} \parr H\{a\}} \parr D \end{array}}{\mathsf{ai}\!\uparrow \dfrac{a \otimes a^{\perp}}{\perp} \parr \begin{array}{c} K\{a^{\perp}\} \parr H\{a\} \parr D \\ \Delta \Vert \\ B \end{array}} \quad \xrightarrow{\text{splitting}} \quad \frac{\begin{array}{c} \overline{\Vert} \\ \left(a \parr K\{a^{\perp}\} \parr D_1\right) \end{array} \otimes \begin{array}{c} \overline{\Vert} \\ \left(a^{\perp} \parr H\{a\} \parr D_2\right) \end{array}}{\left(\left(a \parr K\{a^{\perp}\}\right) \otimes \left(a^{\perp} \parr H\{a\}\right)\right) \parr \begin{array}{c} D_1 \parr D_2 \\ \Vert \\ D \end{array}}$$

$$\boxed{\;\mathsf{ai}\!\downarrow\frac{1}{a \parr a^{\perp}} \qquad \mathsf{s}\,\frac{A \otimes (B \parr C)}{(A \otimes B) \parr C}\;}$$

MLS

If $\overset{\Pi\,\mathsf{MLS}}{(A \otimes B) \parr C}$ then there exist K_A, K_B such that $\begin{array}{c} K_A \parr K_B \\ \| \,\mathsf{MLS} \\ K \end{array}$, $\overset{\Pi\,\mathsf{MLS}}{A \parr K_A}$, $\overset{\Pi\,\mathsf{MLS}}{B \parr K_B}$

If $\overset{\Pi\,\mathsf{MLS}}{a \parr C}$ then $\begin{array}{c} a^{\perp} \\ \| \,\mathsf{MLS} \\ C \end{array}$

Proof is by induction on $(|(A \otimes B) \parr C|,\ n(\Phi))$, where $|A|$ is the number of atom occurrences in A and $n(\Phi)$ is the number of inference steps in Φ.

Cases : (i) $\Phi = \overset{\Phi'\,\Pi}{\left(\rho\,\frac{A'}{A} \otimes B\right) \parr C}$ (ii) $\Phi = \overset{\Phi'\,\Pi}{\left(A \otimes \rho\,\frac{B'}{B}\right) \parr C}$ (iii) $\Phi = \overset{\Phi'\,\Pi}{(A \otimes B) \parr \rho\,\frac{C'}{C}}$

(iv) $\Phi = \overset{\Phi'\,\Pi}{\mathsf{s}\,\frac{A_1 \otimes B_1 \otimes ((A_2 \otimes B_2) \parr C_1)}{(A_1 \otimes A_2 \otimes B_1 \otimes B_2) \parr C_1} \parr C_2}$

(v) $\Phi = \overset{\Phi'\,\Pi}{\frac{((A \otimes B) \parr C_1 \parr C_2) \otimes C_3}{(A \otimes B) \parr C_1 \parr (C_2 \otimes C_3)} \parr C_4}$

If $\begin{array}{c}\Phi\,\Pi\,\mathsf{MLS}\\ (A\otimes B)\parr C\end{array}$ then there exist K_A, K_B such that $\begin{array}{c}K_A\parr K_B\\ \|\,\mathsf{MLS}\\ K\end{array}$, $\begin{array}{c}\Pi\,\mathsf{MLS}\\ A\parr K_A\end{array}$, $\begin{array}{c}\Pi\,\mathsf{MLS}\\ B\parr K_B\end{array}$

(iv) $\Phi = {}_{s}\dfrac{\overset{\Phi'\,\Pi}{A_1\otimes B_1 \otimes ((A_2\otimes B_2)\parr C_1)}}{(A_1\otimes A_2\otimes B_1\otimes B_2)\parr C_1}\parr C_2$

Observation : For all $\rho\frac{A}{B}$ in MLS, $|A|\leqslant|B|$

By induction on Φ', there exist D, E s.t. $\begin{array}{c}D\parr E\\ \|\,\mathsf{MLS}\\ C_2\end{array}$, $\begin{array}{c}\Theta\,\Pi\,\mathsf{MLS}\\ (A_1\otimes B_1)\parr D\end{array}$, $\begin{array}{c}\Omega\,\Pi\,\mathsf{MLS}\\ (A_2\otimes B_2)\parr C_1\parr E\end{array}$

By induction on Θ, there exist D_1, D_2 s.t. $\begin{array}{c}D_1\parr D_2\\ \|\,\mathsf{MLS}\\ D\end{array}$, $\begin{array}{c}\Pi\,\mathsf{MLS}\\ A_1\parr D_1\end{array}$, $\begin{array}{c}\Pi\,\mathsf{MLS}\\ B_1\parr D_2\end{array}$

By induction on Ω, there exist E_1, E_2 s.t. $\begin{array}{c}E_1\parr E_2\\ \|\,\mathsf{MLS}\\ C_1\parr E\end{array}$, $\begin{array}{c}\Pi\,\mathsf{MLS}\\ A_2\parr E_1\end{array}$ $\begin{array}{c}\Pi\,\mathsf{MLS}\\ B_2\parr E_2\end{array}$

Then we take

$K_A = D_1\parr E_1$

$K_2 = D_2\parr E_2$

$$: \begin{array}{c}\begin{array}{c}D_1\parr D_2\\ \|\,\mathsf{MLS}\\ D\end{array}\parr\begin{array}{c}E_1\parr E_2\\ \|\,\mathsf{MLS}\\ C_1\parr E\end{array}\\ \|\,\mathsf{MLS}\\ C_1\parr C_2\end{array} \qquad {}_{2s}\dfrac{\begin{array}{c}\Pi\,\mathsf{MLS}\\ A_1\parr D_1\end{array}\otimes\begin{array}{c}\Pi\,\mathsf{MLS}\\ A_2\parr E_1\end{array}}{(A_1\otimes A_2)\parr D_1\parr E_1} \qquad {}_{2s}\dfrac{\begin{array}{c}\Pi\,\mathsf{MLS}\\ B_1\parr D_2\end{array}\otimes\begin{array}{c}\Pi\,\mathsf{MLS}\\ B_2\parr E_2\end{array}}{(B_1\otimes B_2)\parr D_2\parr E_2}$$

$$\boxed{\mathsf{ai}\!\downarrow\frac{1}{a \invamp a^{\perp}} \qquad \mathsf{s}\,\frac{A \otimes (B \invamp C)}{(A \otimes B) \invamp C}} \quad \mathsf{MLS}$$

$$\Phi \,\Big\Vert\, \mathsf{MLS}$$

$$\left(\mathsf{ai}\!\uparrow\frac{a \otimes a^{\perp}}{\perp} \;\otimes\; B\right) \invamp C$$

$$\boxed{\mathsf{ai}{\downarrow}\,\dfrac{1}{a \parr a^\perp} \qquad \mathsf{s}\,\dfrac{A \otimes (B \parr C)}{(A \otimes B) \parr C}}$$

MLS

$$\begin{array}{c} \Phi \,\Big\|\, \mathsf{MLS} \\ \left(\mathsf{ai}{\uparrow}\,\dfrac{a \otimes a^\perp}{\perp} \otimes B\right) \parr C \end{array} \quad \xrightarrow{\text{splitting on } \Phi} \quad 2\mathsf{s}\,\dfrac{\begin{array}{c} \Psi \,\Big\|\, \mathsf{MLS} \\ \mathsf{ai}{\uparrow}\,\dfrac{a \otimes a^\perp}{\perp} \parr C_1 \end{array} \quad \otimes \quad \begin{array}{c} \Big\|\, \mathsf{MLS} \\ B \parr C_2 \end{array}}{(\perp \otimes B) \parr \begin{array}{c} C_1 \parr C_2 \\ \| \\ C \end{array}}$$

$$\boxed{\ \mathsf{ai}{\downarrow}\,\frac{1}{a \parr a^{\perp}} \qquad \mathsf{s}\,\frac{A \otimes (B \parr C)}{(A \otimes B) \parr C}\ }$$

MLS

$$\begin{array}{c} \Phi \,\Vert\, \mathsf{MLS} \\ \left(\mathsf{ai}{\uparrow}\,\dfrac{a \otimes a^{\perp}}{\perp} \otimes B \right) \parr C \end{array}$$

$\xrightarrow{\text{splitting on } \Phi}$

$$\mathsf{2s}\,\frac{\begin{array}{c} \Psi \,\Vert\, \mathsf{MLS} \\ \mathsf{ai}{\uparrow}\,\dfrac{a \otimes a^{\perp}}{\perp} \parr C_1 \end{array} \otimes \begin{array}{c} \Vert\, \mathsf{MLS} \\ B \parr C_2 \end{array}}{(\perp \otimes B) \parr \begin{array}{c} C_1 \parr C_2 \\ \Vert \\ C \end{array}}$$

splitting on Ψ $\swarrow$

$$\mathsf{2s}\,\frac{\dfrac{\begin{array}{c} \Theta \,\Vert\, \mathsf{MLS} \\ a \parr D_1 \end{array} \otimes \begin{array}{c} \Omega \,\Vert\, \mathsf{MLS} \\ a^{\perp} \parr D_2 \end{array}}{\dfrac{a \otimes a^{\perp}}{\perp} \parr \begin{array}{c} D_1 \parr D_2 \\ \Vert \\ C_1 \end{array}} \otimes \begin{array}{c} \Vert \\ B \parr C_2 \end{array}}{(\perp \otimes B) \parr \begin{array}{c} C_1 \parr C_2 \\ \Vert\, \mathsf{MLS} \\ C \end{array}}$$

$$\boxed{\mathsf{ai}\!\downarrow\frac{1}{a \parr a^\perp} \qquad \mathsf{s}\frac{A\otimes(B\parr C)}{(A\otimes B)\parr C}}$$

MLS

This example shows how we iterate splitting in order to eliminate even a deeply nested cut.

$$\begin{array}{c} \Phi \,\Big\Vert\, \mathsf{MLS} \\ \left(\mathsf{ai}\!\uparrow\frac{a\otimes a^\perp}{\perp}\otimes B\right)\parr C \end{array} \xrightarrow{\text{splitting on } \Phi} \mathsf{2s}\frac{\begin{array}{c}\Psi\,\Big\Vert\,\mathsf{MLS}\\ \mathsf{ai}\!\uparrow\frac{a\otimes a^\perp}{\perp}\parr C_1\end{array} \otimes \begin{array}{c}\Big\Vert\,\mathsf{MLS}\\ B\parr C_2\end{array}}{(\perp\otimes B)\parr\begin{array}{c}C_1\parr C_2\\ \Vert\\ C\end{array}}$$

splitting on Ψ

$$\mathsf{2s}\frac{\begin{array}{c}\dfrac{\begin{array}{c}\Theta\,\Big\Vert\,\mathsf{MLS}\\ a\parr D_1\end{array}\otimes\begin{array}{c}\Omega\,\Big\Vert\,\mathsf{MLS}\\ a^\perp\parr D_2\end{array}}{\dfrac{a\otimes a^\perp}{\perp}\parr\begin{array}{c}D_1\parr D_2\\ \Vert\\ C_1\end{array}}\end{array} \otimes \begin{array}{c}\Big\Vert\\ B\parr C_2\end{array}}{(\perp\otimes B)\parr\begin{array}{c}C_1\parr C_2\\ \Vert\,\mathsf{MLS}\\ C\end{array}} \xrightarrow{\text{atomic splitting on } \Theta \text{ and } \Omega} \mathsf{2s}\frac{\begin{array}{c}\mathsf{ai}\!\downarrow\dfrac{1}{\begin{array}{ccc}a^\perp & & a\\ \Vert\,\mathsf{MLS} & \parr & \Vert\,\mathsf{MLS}\\ D_1 & & D_2\end{array}}\\ \Vert\\ C_1\end{array} \otimes \begin{array}{c}\Big\Vert\\ B\parr C_2\end{array}}{(\perp\otimes B)\parr\begin{array}{c}C_1\parr C_2\\ \Vert\,\mathsf{MLS}\\ C\end{array}}$$

$$\boxed{\text{ai}\!\downarrow\frac{1}{a \parr a^\perp} \qquad \text{s}\,\frac{A \otimes (B \parr C)}{(A \otimes B) \parr C}}$$

MLS

What about another system?

$$\begin{array}{c}\Phi \,\|\, \mathsf{MLS} \\ \left(\text{ai}\!\uparrow\dfrac{a \otimes a^\perp}{\bot} \otimes B\right) \parr C\end{array}$$

splitting on Φ $\longrightarrow$

$$\text{2s}\,\frac{\begin{array}{c}\Psi \,\|\, \mathsf{MLS} \\ \text{ai}\!\uparrow\dfrac{a \otimes a^\perp}{\bot} \parr C_1\end{array} \quad \otimes \quad \begin{array}{c}\|\, \mathsf{MLS} \\ B \parr C_2\end{array}}{(\bot \otimes B) \parr \begin{array}{c}C_1 \parr C_2 \\ \| \\ C\end{array}}$$

splitting on Ψ $\swarrow$

$$\text{2s}\,\frac{\dfrac{\begin{array}{c}\Theta \,\|\, \mathsf{MLS} \\ a \parr D_1\end{array} \otimes \begin{array}{c}\Omega \,\|\, \mathsf{MLS} \\ a^\perp \parr D_2\end{array}}{\dfrac{a \otimes a^\perp}{\bot} \parr \begin{array}{c}D_1 \parr D_2 \\ \| \\ C_1\end{array}} \quad \otimes \quad \begin{array}{c}\| \\ B \parr C_2\end{array}}{(\bot \otimes B) \parr \begin{array}{c}C_1 \parr C_2 \\ \|\, \mathsf{MLS} \\ C\end{array}}$$

atomic splitting on Θ and Ω $\longrightarrow$

$$\text{2s}\,\frac{\text{ai}\!\downarrow\dfrac{1}{\begin{array}{c}\begin{array}{c}a^\perp \\ \|\, \mathsf{MLS} \\ D_1\end{array} \parr \begin{array}{c}a \\ \|\, \mathsf{MLS} \\ D_2\end{array} \\ \| \\ C_1\end{array}} \quad \otimes \quad \begin{array}{c}\| \\ B \parr C_2\end{array}}{(\bot \otimes B) \parr \begin{array}{c}C_1 \parr C_2 \\ \|\, \mathsf{MLS} \\ C\end{array}}$$

$$\mathsf{ai}{\downarrow}\frac{1}{a \parr a^{\perp}} \qquad \mathsf{s}\frac{A \otimes (B \parr C)}{(A \otimes B) \parr C} \qquad \mathsf{q}{\downarrow}\frac{(A \parr B) \lhd (C \parr D)}{(A \lhd C) \parr (B \lhd D)}$$

BV

What about another system?

$$\begin{array}{c} \Phi \,\Vert\, \mathsf{BV} \\ \left(\mathsf{ai}{\uparrow}\dfrac{a \otimes a^{\perp}}{\perp} \lhd B \right) \parr C \end{array}$$

$\xrightarrow{\text{splitting on } \Phi}$

$$\begin{array}{c} \Psi \,\Vert\, \mathsf{BV} \\ \mathsf{2s}\dfrac{\left[\mathsf{ai}{\uparrow}\dfrac{a \otimes a^{\perp}}{\perp} \parr C_1\right] \lhd \left[\begin{array}{c} \Vert\, \mathsf{BV} \\ B \parr C_2 \end{array}\right]}{(\perp \lhd B) \parr \begin{array}{c} C_1 \lhd C_2 \\ \Vert\, \mathsf{BV} \\ C \end{array}} \end{array}$$

splitting on Ψ ↙

$$\mathsf{2s}\dfrac{\left[\dfrac{\begin{array}{c}\Theta \,\Vert\, \mathsf{BV} \\ a \parr D_1\end{array} \otimes \begin{array}{c}\Omega \,\Vert\, \mathsf{BV} \\ a^{\perp} \parr D_2\end{array}}{\dfrac{a \otimes a^{\perp}}{\perp} \parr \begin{array}{c} D_1 \parr D_2 \\ \Vert\, \mathsf{BV} \\ C_1 \end{array}}\right] \lhd \left[\begin{array}{c} \Vert\, \mathsf{BV} \\ B \parr C_2 \end{array}\right]}{(\perp \lhd B) \parr \begin{array}{c} C_1 \lhd C_2 \\ \Vert\, \mathsf{BV} \\ C \end{array}}$$

$\xrightarrow{\text{atomic splitting on } \Theta \text{ and } \Omega}$

$$\mathsf{2s}\dfrac{\left[\begin{array}{c} \mathsf{ai}{\downarrow}\dfrac{1}{\begin{array}{c} a^{\perp} \\ \Vert\, \mathsf{BV} \\ D_1 \end{array} \parr \begin{array}{c} a \\ \Vert\, \mathsf{BV} \\ D_2 \end{array}} \\ \Vert \\ C_1 \end{array}\right] \lhd \left[\begin{array}{c} \Vert\, \mathsf{BV} \\ B \parr C_2 \end{array}\right]}{(\perp \lhd B) \parr \begin{array}{c} C_1 \lhd C_2 \\ \Vert\, \mathsf{BV} \\ C \end{array}}$$

$$\text{ai}\downarrow\frac{1}{a \parr a^{\perp}} \qquad \mathsf{s}\frac{A\otimes(B\parr C)}{(A\otimes B)\parr C} \qquad \mathsf{q}\downarrow\frac{(A\parr B)\lhd(C\parr D)}{(A\lhd C)\parr(B\lhd D)}$$

BV

If $\begin{array}{c}\Phi\,\Pi\,\mathsf{BV}\\ (A\otimes B)\parr C\end{array}$ then there exist K_A, K_B such that $\begin{array}{c}K_A\parr K_B\\ \|\,\mathsf{BV}\\ K\end{array}$, $\begin{array}{c}\Pi\,\mathsf{BV}\\ A\parr K_A\end{array}$, $\begin{array}{c}\Pi\,\mathsf{BV}\\ B\parr K_B\end{array}$

If $\begin{array}{c}\Phi\,\Pi\,\mathsf{BV}\\ (A\lhd B)\parr C\end{array}$ then there exist K_A, K_B such that $\begin{array}{c}K_A\lhd K_B\\ \|\,\mathsf{BV}\\ K\end{array}$, $\begin{array}{c}\Pi\,\mathsf{BV}\\ A\parr K_A\end{array}$, $\begin{array}{c}\Pi\,\mathsf{BV}\\ B\parr K_B\end{array}$

If $\begin{array}{c}\Pi\,\mathsf{BV}\\ a\parr C\end{array}$ then $\begin{array}{c}a^{\perp}\\ \|\,\mathsf{BV}\\ C\end{array}$

This is the theorem that we need to prove in order to extend the splitting technique to system BV+cut

Identifying and proving the cases left as an exercise

Cases : (i) $\Phi = \begin{array}{c}\Phi'\,\Pi\\ \mathsf{q}\downarrow\dfrac{(A_1\parr C_1)\lhd((A_2\lhd B)\parr C_2)}{(A_1\lhd A_2\lhd B)\parr(C_1\lhd C_2)}\end{array}\parr C_3$

(ii) $\Phi = ?$

…

$$\mathsf{ai}\!\downarrow\frac{1}{a \parr a^{\perp}} \qquad \mathsf{s}\,\frac{A \otimes (B \parr C)}{(A \otimes B) \parr C} \qquad \mathsf{q}\!\downarrow\frac{(A \parr B) \triangleleft (C \parr D)}{(A \triangleleft C) \parr (B \triangleleft D)} \qquad \mathsf{w}\!\downarrow\frac{\perp}{?A} \qquad \mathsf{g}\!\downarrow\frac{??A}{?A} \qquad \mathsf{b}\!\downarrow\frac{?A \parr A}{?A} \qquad \mathsf{p}\!\downarrow\frac{!(A \parr B)}{!A \parr ?B} \qquad \mathsf{e}\!\downarrow\frac{1}{!1}$$

NEL

$$\Phi \,\Big\Vert\, \mathsf{NEL}$$

$$!\left(\mathsf{ai}\!\uparrow\frac{a \otimes a^{\perp}}{\perp}\right) \parr C$$

What about another system?

$$\mathsf{ai}{\downarrow}\frac{1}{a \parr a^{\perp}} \qquad \mathsf{s}\frac{A \otimes (B \parr C)}{(A \otimes B) \parr C} \qquad \mathsf{q}{\downarrow}\frac{(A \parr B) \lhd (C \parr D)}{(A \lhd C) \parr (B \lhd D)} \qquad \mathsf{w}{\downarrow}\frac{\perp}{?A} \qquad \mathsf{g}{\downarrow}\frac{??A}{?A} \qquad \mathsf{b}{\downarrow}\frac{?A \parr A}{?A} \qquad \mathsf{p}{\downarrow}\frac{!(A \parr B)}{!A \parr ?B} \qquad \mathsf{e}{\downarrow}\frac{1}{!1}$$

NEL

$$\begin{array}{c} \Phi \,\big\|\, \mathsf{NEL} \\ !\left(\mathsf{ai}{\uparrow}\dfrac{a \otimes a^{\perp}}{\perp}\right) \parr C \end{array}$$

splitting on Φ →

$$\begin{array}{l}
!\left(\begin{array}{c} \Psi \,\big\|\, \mathsf{NEL} \\ \mathsf{i}{\uparrow}\dfrac{a \otimes a^{\perp}}{\perp} \parr C_1 \parr \cdots \parr C_n \end{array}\right) \\
\mathsf{p}{\downarrow}\overline{\qquad\qquad\qquad\qquad\qquad\qquad} \\
!(\perp \parr C_1 \parr \cdots \parr C_{n-1}) \parr ?C_n \\
\mathsf{p}{\downarrow}\overline{\qquad\qquad\qquad\qquad\qquad\qquad} \\
\qquad\qquad\vdots \\
\mathsf{p}{\downarrow}\overline{\qquad\qquad\qquad\qquad\qquad\qquad} \\
!\perp \parr \left(\begin{array}{c} ?C_1 \parr \cdots \parr ?C_n \\ \big\|\, \mathsf{NEL} \\ C \end{array}\right)
\end{array}$$

What about another system?

$$\mathsf{ai}{\downarrow}\,\frac{1}{a \parr a^{\perp}} \qquad \mathsf{s}\,\frac{A \otimes (B \parr C)}{(A \otimes B) \parr C} \qquad \mathsf{q}{\downarrow}\,\frac{(A \parr B) \lhd (C \parr D)}{(A \lhd C) \parr (B \lhd D)} \qquad \mathsf{w}{\downarrow}\,\frac{\perp}{?A} \qquad \mathsf{g}{\downarrow}\,\frac{??A}{?A} \qquad \mathsf{b}{\downarrow}\,\frac{?A \parr A}{?A} \qquad \mathsf{p}{\downarrow}\,\frac{!(A \parr B)}{!A \parr ?B} \qquad \mathsf{e}{\downarrow}\,\frac{1}{!1}$$

NEL

$$\begin{array}{c} \Phi \,\Vert\, \mathsf{NEL} \\ !\left(\mathsf{ai}{\uparrow}\,\frac{a \otimes a^{\perp}}{\perp}\right) \parr C \end{array}$$

splitting on Φ →

$$\begin{array}{c} !\left(\begin{array}{c} \Psi \,\Vert\, \mathsf{NEL} \\ \mathsf{i}{\uparrow}\,\frac{a \otimes a^{\perp}}{\perp} \parr C_1 \parr \cdots \parr C_n \end{array}\right) \\ \hline \mathsf{p}{\downarrow}\quad !(\perp \parr C_1 \parr \cdots \parr C_{n-1}) \parr ?C_n \\ \hline \mathsf{p}{\downarrow}\quad \vdots \\ \hline \mathsf{p}{\downarrow}\quad !\perp \parr \left(\begin{array}{c} ?C_1 \parr \cdots \parr ?C_n \\ \Vert\, \mathsf{NEL} \\ C \end{array}\right) \end{array}$$

splitting on Ψ and atomic splitting →

$$\begin{array}{c} \mathsf{e}{\downarrow}\,\dfrac{1}{!\left(\begin{array}{c} \mathsf{ai}{\downarrow}\,\dfrac{1}{\begin{array}{ccc} a^{\perp} & & a \\ \Vert\, \mathsf{NEL} & \parr & \Vert\, \mathsf{NEL} \\ D_1 & & D_2 \end{array}} \\ \Vert \\ = \dfrac{C_1 \parr \cdots \parr C_n}{\perp \parr C_1 \parr \cdots \parr C_n} \end{array}\right)} \\ \Vert\, \mathsf{p}{\downarrow} \\ !\perp \parr \left(\begin{array}{c} ?C_1 \parr \cdots \parr ?C_n \\ \Vert\, \mathsf{NEL} \\ C \end{array}\right) \end{array}$$

What about another system?

$$\mathsf{ai}\!\downarrow \frac{1}{a \parr a^{\perp}} \qquad \mathsf{s}\,\frac{A \otimes (B \parr C)}{(A \otimes B) \parr C} \qquad \mathsf{q}\!\downarrow \frac{(A \parr B) \lhd (C \parr D)}{(A \lhd C) \parr (B \lhd D)} \qquad \mathsf{w}\!\downarrow \frac{\perp}{?A} \qquad \mathsf{g}\!\downarrow \frac{??A}{?A} \qquad \mathsf{b}\!\downarrow \frac{?A \parr A}{?A} \qquad \mathsf{p}\!\downarrow \frac{!(A \parr B)}{!A \parr ?B} \qquad \mathsf{e}\!\downarrow \frac{1}{!1}$$

NEL

If $\overset{\Phi\,\Pi_{\mathsf{NEL}}}{(A \otimes B) \parr C}$ then there exist K_A, K_B such that $\begin{array}{c} K_A \parr K_B \\ \| \mathsf{NEL} \\ K \end{array}$, $\overset{\Pi_{\mathsf{NEL}}}{A \parr K_A}$, $\overset{\Pi_{\mathsf{NEL}}}{B \parr K_B}$

If $\overset{\Phi\,\Pi_{\mathsf{NEL}}}{(A \lhd B) \parr C}$ then there exist K_A, K_B such that $\begin{array}{c} K_A \lhd K_B \\ \| \mathsf{NEL} \\ K \end{array}$, $\overset{\Pi_{\mathsf{NEL}}}{A \parr K_A}$, $\overset{\Pi_{\mathsf{NEL}}}{B \parr K_B}$

If $\overset{\Phi\,\Pi_{\mathsf{NEL}}}{!A \parr C}$ then there exist $K_1, \ldots, K_n$ such that $\begin{array}{c} ?K_1 \parr \cdots \parr ?K_n \\ \| \mathsf{NEL} \\ C \end{array}$, $\overset{\Pi_{\mathsf{NEL}}}{A \parr K_1 \parr \cdots \parr K_n}$

If $\overset{\Phi\,\Pi_{\mathsf{NEL}}}{?A \parr C}$ then there exists K such that $\begin{array}{c} !K \\ \| \mathsf{NEL} \\ C \end{array}$, $\overset{\Pi_{\mathsf{NEL}}}{A \parr K}$

If $\overset{\Pi_{\mathsf{NEL}}}{a \parr C}$ then $\begin{array}{c} a^{\perp} \\ \| \mathsf{NEL} \\ C \end{array}$

$$
\mathsf{ai}\!\downarrow\frac{1}{a \parr a^{\perp}}
\qquad
\mathsf{s}\frac{A \otimes (B \parr C)}{(A \otimes B) \parr C}
\qquad
\mathsf{q}\!\downarrow\frac{(A \parr B) \lhd (C \parr D)}{(A \lhd C) \parr (B \lhd D)}
\qquad
\mathsf{w}\!\downarrow\frac{\perp}{?A}
\qquad
\mathsf{g}\!\downarrow\frac{??A}{?A}
\qquad
\mathsf{b}\!\downarrow\frac{?A \parr A}{?A}
\qquad
\mathsf{p}\!\downarrow\frac{!(A \parr B)}{!A \parr ?B}
\qquad
\mathsf{e}\!\downarrow\frac{1}{!1}
$$

NEL

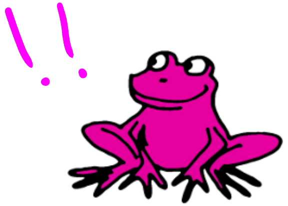
If $\begin{array}{c} \Phi\,\Pi\,\mathsf{NEL} \\ (A \otimes B) \parr C \end{array}$ then there exist K_A, K_B such that $\begin{array}{c} K_A \parr K_B \\ \Vert\,\mathsf{NEL} \\ K \end{array}$, $\begin{array}{c} \Pi\,\mathsf{NEL} \\ A \parr K_A \end{array}$, $\begin{array}{c} \Pi\,\mathsf{NEL} \\ B \parr K_B \end{array}$

If $\begin{array}{c} \Phi\,\Pi\,\mathsf{NEL} \\ (A \lhd B) \parr C \end{array}$ then there exist K_A, K_B such that $\begin{array}{c} K_A \lhd K_B \\ \Vert\,\mathsf{NEL} \\ K \end{array}$, $\begin{array}{c} \Pi\,\mathsf{NEL} \\ A \parr K_A \end{array}$, $\begin{array}{c} \Pi\,\mathsf{NEL} \\ B \parr K_B \end{array}$

If $\begin{array}{c} \Phi\,\Pi\,\mathsf{NEL} \\ !A \parr C \end{array}$ then there exist $K_1, \ldots, K_n$ such that $\begin{array}{c} ?K_1 \parr \cdots \parr ?K_n \\ \Vert\,\mathsf{NEL} \\ C \end{array}$, $\begin{array}{c} \Pi\,\mathsf{NEL} \\ A \parr K_1 \parr \cdots \parr K_n \end{array}$

If $\begin{array}{c} \Phi\,\Pi\,\mathsf{NEL} \\ ?A \parr C \end{array}$ then there exists K such that $\begin{array}{c} !K \\ \Vert\,\mathsf{NEL} \\ C \end{array}$, $\begin{array}{c} \Pi\,\mathsf{NEL} \\ A \parr K \end{array}$

If $\begin{array}{c} \Pi\,\mathsf{NEL} \\ a \parr C \end{array}$ then $\begin{array}{c} a^{\perp} \\ \Vert\,\mathsf{NEL} \\ C \end{array}$



This looks like the theorem we want, but it cannot be proven

$$\mathsf{ai}{\downarrow}\dfrac{1}{a \parr a^{\bot}} \qquad \mathsf{s}\dfrac{A \otimes (B \parr C)}{(A \otimes B) \parr C} \qquad \mathsf{q}{\downarrow}\dfrac{(A \parr B) \lhd (C \parr D)}{(A \lhd C) \parr (B \lhd D)} \qquad \mathsf{w}{\downarrow}\dfrac{\bot}{?A} \qquad \mathsf{g}{\downarrow}\dfrac{??A}{?A} \qquad \mathsf{b}{\downarrow}\dfrac{?A \parr A}{?A} \qquad \mathsf{p}{\downarrow}\dfrac{!(A \parr B)}{!A \parr ?B} \qquad \mathsf{e}{\downarrow}\dfrac{1}{!1}$$

NEL

(i)

$$\begin{array}{c} \Phi' \, \Pi \\ \mathsf{w}{\downarrow}\dfrac{\bot}{?A} \parr C \end{array}$$

$$\begin{array}{c} !K \\ \Vert \\ C \end{array} \; , \quad \begin{array}{c} \Pi \\ A \parr K \end{array}$$

(ii)

$$\begin{array}{c} \Phi' \, \Pi \\ \mathsf{b}{\downarrow}\dfrac{?A \parr A}{?A} \parr C \end{array}$$

IH not satisfied

(iii)

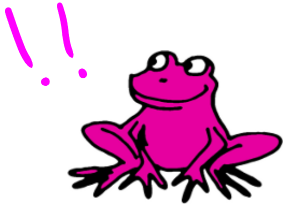
$$\begin{array}{c} \Phi' \, \Pi \\ \mathsf{g}{\downarrow}\dfrac{??A}{?A} \parr C \end{array}$$

Breaks previous cases

If $\begin{array}{c} \Phi \, \Pi_{\mathsf{NEL}} \\ ?A \parr C \end{array}$ then there exists K such that $\begin{array}{c} !K \\ \Vert_{\mathsf{NEL}} \\ C \end{array}$, $\begin{array}{c} \Pi_{\mathsf{NEL}} \\ A \parr K \end{array}$

Proof is by induction on

$$(|(A \otimes B) \parr C| \, , \; n(\Phi))$$



$$\boxed{\mathsf{ai}{\downarrow}\,\frac{1}{a \parr a^{\perp}} \quad \mathsf{s}\,\frac{A \otimes (B \parr C)}{(A \otimes B) \parr C} \quad \mathsf{q}{\downarrow}\,\frac{(A \parr B) \triangleleft (C \parr D)}{(A \triangleleft C) \parr (B \triangleleft D)} \quad \mathsf{w}{\downarrow}\,\frac{\perp}{?A} \quad \mathsf{g}{\downarrow}\,\frac{??A}{?A} \quad \mathsf{b}{\downarrow}\,\frac{?A \parr A}{?A} \quad \mathsf{p}{\downarrow}\,\frac{!(A \parr B)}{!A \parr ?B} \quad \mathsf{e}{\downarrow}\,\frac{1}{!1}}$$

NELc

If $\begin{array}{c} \Phi\,\Pi_{\mathsf{NELc}} \\ (A \otimes B) \parr C \end{array}$ then there exist K_A, K_B such that $\begin{array}{c} K_A \parr K_B \\ \| \mathsf{NELc} \\ K \end{array}$, $\begin{array}{c} \Pi_{\mathsf{NELc}} \\ A \parr K_A \end{array}$, $\begin{array}{c} \Pi_{\mathsf{NELc}} \\ B \parr K_B \end{array}$

If $\begin{array}{c} \Phi\,\Pi_{\mathsf{NELc}} \\ (A \triangleleft B) \parr C \end{array}$ then there exist K_A, K_B such that $\begin{array}{c} K_A \triangleleft K_B \\ \| \mathsf{NELc} \\ K \end{array}$, $\begin{array}{c} \Pi_{\mathsf{NELc}} \\ A \parr K_A \end{array}$, $\begin{array}{c} \Pi_{\mathsf{NELc}} \\ B \parr K_B \end{array}$

If $\begin{array}{c} \Phi\,\Pi_{\mathsf{NELc}} \\ !A \parr C \end{array}$ then there exist $K_1, \dots, K_n$ such that $\begin{array}{c} ?K_1 \parr \dots \parr ?K_n \\ \| \mathsf{NELc} \\ C \end{array}$, $\begin{array}{c} \Pi_{\mathsf{NELc}} \\ A \parr K_1 \parr \dots \parr K_n \end{array}$

If $\begin{array}{c} \Phi\,\Pi_{\mathsf{NELc}} \\ ?A \parr C \end{array}$ then there exists K such that $\begin{array}{c} !K \\ \| \mathsf{NELc} \\ C \end{array}$, $\begin{array}{c} \Pi_{\mathsf{NELc}} \\ A \parr K \end{array}$

If $\begin{array}{c} \Pi_{\mathsf{NELc}} \\ a \parr C \end{array}$ then $\begin{array}{c} a^{\perp} \\ \| \mathsf{NELc} \\ C \end{array}$

If we only have the 'core' rules then the splitting theorem is provable

$$\mathsf{ai}\!\downarrow\frac{1}{a \parr a^{\perp}} \qquad \mathsf{s}\,\frac{A \otimes (B \parr C)}{(A \otimes B) \parr C} \qquad \mathsf{q}\!\downarrow\frac{(A \parr B) \triangleleft (C \parr D)}{(A \triangleleft C) \parr (B \triangleleft D)} \qquad \mathsf{w}\!\downarrow\frac{\perp}{?A} \qquad \mathsf{g}\!\downarrow\frac{??A}{?A} \qquad \mathsf{b}\!\downarrow\frac{?A \parr A}{?A} \qquad \mathsf{p}\!\downarrow\frac{!(A \parr B)}{!A \parr ?B} \qquad \mathsf{e}\!\downarrow\frac{1}{!1}$$

NELc

$$\text{If} \quad \begin{array}{l} A \\ \Vert_{\mathsf{NEL}} \\ B \end{array} \quad \text{then} \quad \begin{array}{l} A \\ \Vert\ w\uparrow,\ g\uparrow,\ b\uparrow \\ A' \\ \Vert_{\mathsf{NELc}} \\ B' \\ \Vert\ w\downarrow,\ g\downarrow,\ b\downarrow \\ B \end{array}$$

Derivations can be **decomposed** into phases

$$\text{ai}\!\downarrow\frac{1}{a \parr a^{\perp}} \quad \text{s}\frac{A \otimes (B \parr C)}{(A \otimes B) \parr C} \quad \text{q}\!\downarrow\frac{(A \parr B) \lhd (C \parr D)}{(A \lhd C) \parr (B \lhd D)} \quad \text{w}\!\downarrow\frac{\perp}{?A} \quad \text{g}\!\downarrow\frac{??A}{?A} \quad \text{b}\!\downarrow\frac{?A \parr A}{?A} \quad \text{p}\!\downarrow\frac{!(A \parr B)}{!A \parr ?B} \quad \text{e}\!\downarrow\frac{1}{!1}$$

NELc

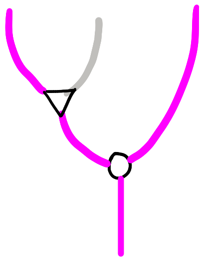
$$\text{If} \quad \begin{array}{c} A \\ \Vert \text{NEL} \\ B \end{array} \quad \text{then} \quad \begin{array}{c} A \\ \Vert \text{w}\!\uparrow, \text{g}\!\uparrow, \text{b}\!\uparrow \\ A' \\ \Vert \text{NELc} \\ B' \\ \Vert \text{w}\!\downarrow, \text{g}\!\downarrow, \text{b}\!\downarrow \\ B \end{array}$$

$$\text{p}\!\uparrow\frac{\text{b}\!\downarrow\dfrac{?A \parr A}{?A} \otimes\ !B}{?(A \otimes B)} \quad \searrow$$

$$\text{b}\!\downarrow\frac{\text{2s}\dfrac{(?A \parr A) \otimes \text{b}\!\uparrow\dfrac{!B}{!B \otimes B}}{\text{p}\!\uparrow\dfrac{?A \otimes !B}{?(A \otimes B)} \parr (A \otimes B)}}{?(A \otimes B)}$$

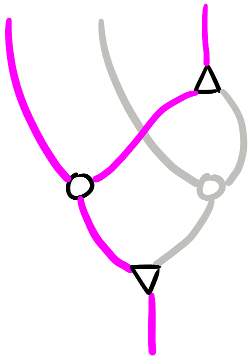
Derivations can be **decomposed** into phases

$$p\!\uparrow\dfrac{b\!\downarrow\dfrac{?A \parr A}{?A} \quad \otimes \quad !B}{?(A \otimes B)}$$

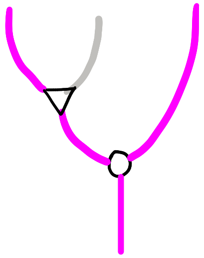


$$\longrightarrow$$

$$b\!\downarrow\dfrac{2s\dfrac{(?A \parr A) \quad \otimes \quad b\!\uparrow\dfrac{!B}{!B \otimes B}}{p\!\uparrow\dfrac{?A \otimes !B}{?(A \otimes B)} \quad \parr \quad (A \otimes B)}}{?(A \otimes B)}$$

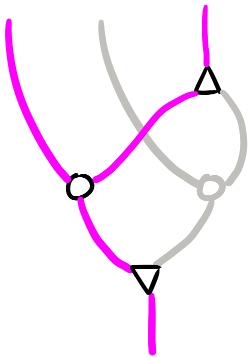


$$\mathsf{p}{\uparrow}\,\dfrac{\mathsf{b}{\downarrow}\,\dfrac{?A \parr A}{?A} \quad \otimes \quad !B}{?(A \otimes B)}$$

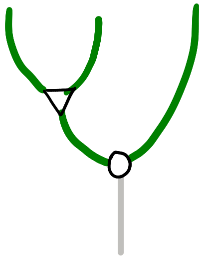


→

$$\mathsf{b}{\downarrow}\,\dfrac{2\mathsf{s}\,\dfrac{(?A \parr A) \quad \otimes \quad \mathsf{b}{\uparrow}\,\dfrac{!B}{!B \otimes B}}{\mathsf{p}{\uparrow}\,\dfrac{?A \otimes !B}{?(A \otimes B)} \quad \parr \quad (A \otimes B)}}{?(A \otimes B)}$$



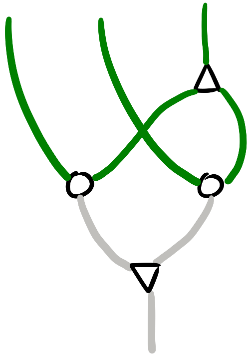
$$\mathsf{ai}{\uparrow}\,\dfrac{\mathsf{ac}{\downarrow}\,\dfrac{a \vee a}{a} \quad \wedge \quad a^{\perp}}{\perp}$$

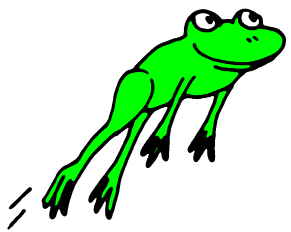


→

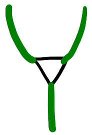
$$\mathsf{c}{\downarrow}\,\dfrac{2\mathsf{s}\,\dfrac{(a \vee a) \quad \wedge \quad \mathsf{ac}{\uparrow}\,\dfrac{a^{\perp}}{a^{\perp} \wedge a^{\perp}}}{\mathsf{ai}{\uparrow}\,\dfrac{a \wedge a^{\perp}}{\perp} \quad \vee \quad \mathsf{ai}{\uparrow}\,\dfrac{a \wedge a^{\perp}}{\perp}}}{\perp}$$

$=$ or c↓

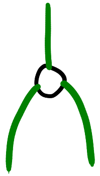




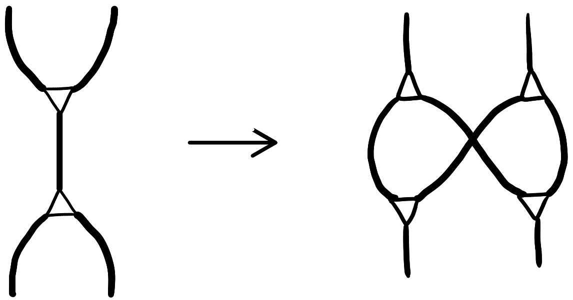
Regularity of inference rules and normalisation procedures

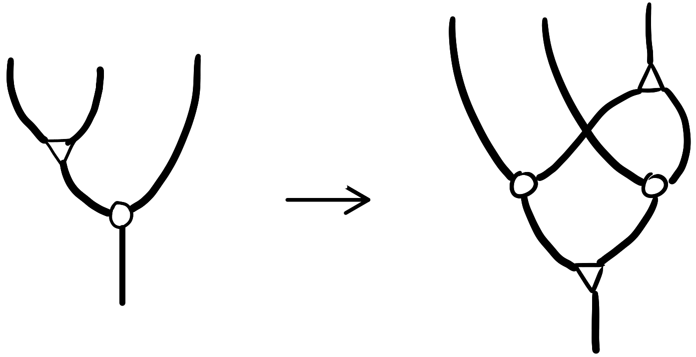


contraction-like rules $$m^{\beta}_{\alpha}\,\frac{(A \,\alpha\, B)\ \beta\ (C \,\alpha\, D)}{(A \,\beta\, C)\ \alpha\ (B \,\beta\, D)}$$



identity-like rules $$s^{\beta}_{\alpha}\,\frac{(A \,\alpha\, B)\ \beta^{\wedge}\ (C \,\alpha\, D)}{(A \,\beta^{\wedge}\, C)\ \alpha\ (B \,\beta^{\vee}\, D)}$$

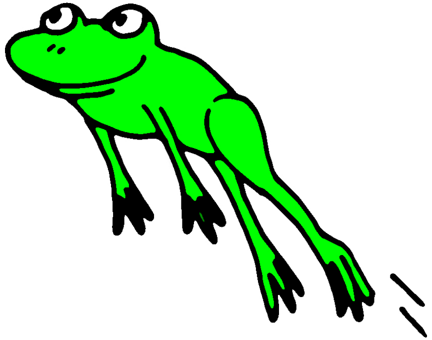




Summary

We introduced open deduction and saw how symmetry, atomicity, and low bureaucracy lead to good normalisation

Things we missed : proof compression mechanisms, combinatorial proofs, subatomic logic, ...



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